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Analysis and Computer Simulation of Urban Cluster Distributions †

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1 Introduction: Self-Organization in the Evolution of Urban Structures

1.1 The Approach of Self-Organization

In recent years, the development of the interdisciplinary field "science of complexity" has lead to the insight, that complex dynamic processes may result from simple rules. In order to reveal those simple mechanisms, it is important to find a level of description, which on one hand considers some specific features of the system, but on the other hand is not flooded with microscopic details. Such a description should include a dynamics which is suitable to reflect the origination of new qualities, the unfolding of complexity, and the limited predictability of the future.

Today's self-organization theory provides different approaches which fulfill these criteria. Numerous models, e.g. in physics, chemistry, molecular biology, but also in economy or sociology, have demonstrated that even simple dynamic assumptions may generate qualitatively new features of the systems, which, on the macroscopic level, are recognized as characteristic "patterns".

Within the framework of self-organization theory, a scenario can be derived which describes this process as an interplay of local interactions and global boundary conditions. Here, the acting elements, due to their eigen dynamics and their specific interactions, commonly create an order parameter, which in turn feeds back to the further evolution of the system and partially "enslaves" its possibilities (to quote a term introduced by Haken, 1978). Additionally, evolving systems effect their environment, which in turn also influences their further development.

The self-organized structure formation can be considered as the opposite to a hierarchical design of structures, which basically proceeds *from top down to bottom*: here, structures are *originated*

†The paper appeared in the version of 1996.

bottom up, leading to an emerging hierarchy, where the structure of the "higher" level appears as a new quality of the system, which in turn feeds back to the "lower" levels.

In this chapter, the approach of self-organization should be applied to the field of *urban structure formation* on a very basic level, which only considers structural or morphological features of urban aggregates. However, if we can obtain very simple rules for the self-organization of urban aggregations, this might shed some new light on the established ideas about hierarchical planning of urban settlements.

1.2 Evolution of Urban Structures: Self-Organization *and* Planning

The formation of urban structures, like settlements, trail systems, transport and supply networks, shows, from a structural perspective, significant analogies to phase transitions, known from physics, like cluster formation, aggregation and percolation phenomena, but also to self-organizing processes, in which new system qualities emerge due to the dynamic interaction of sub-entities.

Usually, one is convinced, that the development of urban areas is determined by numerous factors, difficult to grasp, such as cultural, sociological, economic, political, ecological etc. ingredients. On the other hand, at a certain level of abstraction, one can also find many common features between different urban structures, even from far distant regions, as well as between urban and natural structures [Batty, Longley, 1994; Becker *et al.*, 1994, Schaur, 1991]. This does not only hold for urban aggregates (cf. Sect. 2.1); but also for trail systems, where structural analogies between human-made and animal trail systems, and also to natural supply systems, like vein networks in leaves, have been detected [Schaur, 1991].

These investigations lead us to the hypothesis, that there is a common level to describe the formation of these different structures. From our perspective, the basic dynamics of structure formation creates these analogies, regardless of the different sub-entities involved, and we are convinced that the theory of self-organization provides a suitable quantitative approach to it [Helbing *et al.*, 1994].

In order to specify self-organizing processes in the field of urban structure formation, a dialog is needed between natural science, familiar with the quantitatve aspects of the self-organization theory, and urban planning, familiar with the problems in urban evolution. This dialog should have two aims: First, it should help to overcome the gap between natural science and living science (i) by providing methods of the natural sciences, which could be adapted for solving problems in the social or living sciences *(transfer of methods)*, and (ii) by increasing among natural scientists the sensitivity for problems in the social sciences *(awareness of problems)*. Secondly, the dialog should lead to a new insight into the development of urban settlements by incorporating the ideas of self-organization. Hence, instead of a hierachical planning of details, there could be a quest for the system-immanent forces which allow the generation of urban structures in a new way.

This turn away from total planning, is not only a question of developing new theories, it is challenged by the pressure resulting from the population increase in the developing countries. This population growth leads to an urban growth which cannot be controlled or regulated any further, hence established methods of urban planning become an illusion when confronted with these realities. What could be an alternative? To give new power to the structure generating forces existing also in urban systems, but to determine the conditions (e.g. the topological conditions) which set limits to the self-organization process. This, however, needs a better understanding of how urban self-organization proceeds, and a quantitative description to simulate the processes involved.

2 Analysis of the Settlement Density of Urban Clusters

2.1 Problems and Previous Investigations

From the perspective of physics, urban aggregations, like cities with the surrounding satelite towns and commercial areas, or large agglomerations of mega-cities, can be described as a special kind of *clusters* on a two-dimensional surface. Using a black-and-white reduction, these clusters represent the build-up area (black pixel = occupied site, build up, white pixel = empty site, no building). Here the different types of buildings, e.g. a skyscraper compared to a garage, is not considered. Based on these "black-and-white" maps, the structural featurs of the urban clusters can be investigated in a quantitative analysis.

In order to describe the urban clusters, some questions arise in relation to the methods already established in theoretical physics:

- How can urban clusters be characterized in a quantitative manner? Can physical measures be transfered to describe these structures?
- Do "universal" measures exist for the description of urban clusters, which do not depend on the real topology of the landscape, or on cultural specifications?
- How do urban clusters grow? What is the difference in the dynamics of urban clusters compared to physical cluster growth?
- Do critical values exist, at which the growth of urban clusters stops? Are there "screening effects" for urban growth initiated by other clusters in the neighborhood?
- What are the conditions for a merger of neighboring urban clusters (coalescence)? Contrary, under which conditions does a coexistence of neighboring clusters occur?

Recently, for a description of the structural features and the kinetics of urban clusters, *fractal concepts* have been used:

- *Calculation of the fractal dimension*
  The fractal dimension is one possible measure to describe the build-up density, with different ways of determination [Frankhauser, 1991, 1992, 1994]. The existence of fractal properties over several spatial scales indicates a hierarchical organization of the urban aggregates.

- *Calculation of the ratio of settlement area and circumference*
  The quantitative analysis of 60 metropolitan areas all over the world has shown a linear relation between settlement area and total circumference [Frankhauser, Sadler, 1992], contrary to compact two-dimensional geometrical objects, which follow a square relation. The linear relation typical for urban agglomarations is related to the fractal properties and seems to be an universal feature of these aggregates.

- *Fractal Cluster Growth*
  Previous attempts to simulate the growth of urban clusters by means of fractal aggregation mechanisms, are mainly based on DLA- or DBM models (DLA - diffusion limited aggregation, DBM - dielectrical breakdown model) [Batty, 1991]. In these models, the particles for the cluster growth are released in a larger distance from the cluster, then diffuse and aggregate with certain probabilities to the existing cluster. In the DLA model, constant probabilities for the attachment of particles exist, which lead to the dendritic growth typical for fractals. In the DBM model, these probabilities could be determined due to an additional potential condition, which basically considers the occupation of the neighboring sites.

With respect to the kinetics of urban growth, the existing results can only partially convince. Namely, DLA models only produce connected clusters, with the largest growth potential at the tips of the cluster, far away from the center. Also, there is no optical similarity between the strongly dendritic structures and urban clusters. In DBM models, the compactness of the clusters can be changed with an additional parameter entering the aggregation probabilities [Batty, 1991]. However, it is not considered that the compactness of urban clusters change in the course of time. During an early stage, these clusters are rather compact (similar to EDEN clusters), only in later stages they change their shape towards a dendritic morphology. Moreover, also in DBM models only connected clusters occur, and phenomena specific for urban agglomerations, such as the burn-out of urban growth, the emergence of new growth zones, or the coexistence of urban clusters, cannot be sufficiently described within this approach.

There is another class of kinetic models for urban growth, based on cellular automata which have been mainly developed by WHITE, ENGELEN (1993, 1994, 1996). Here, the focus is on the simulation of "urban land-use patterns". Accordingly, the urban growth is more or less determined by an economically based utility function [Weidlich, 1996], and not by physical forces which govern the structure formation.

In this chapter, the structural aspects of urban cluster growth an their relation to physical models are investigated. In Sect. 2.2. and 2.3. we analyze the spatial distribution of the settlement area. In Sect. 3, we focus on the fact that urban aggregates consist of many smaller clusters which are not merged, and show results of the analysis (Sect. 3.1.) and the computer simulations (Sect. 3.2) of the cluster distribution of urban aggregates. Finally, in Sect. 4 we give an outlook to a more detailed model of urban aggregation, based on reaction-diffusion systems. This model should be capable to describe different effects of urban growth, like the burn-out of growth, or the shift of growth zones.

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2.2 Evaluation of Size and Center of Gravity of the Urban Cluster of Berlin

In this section, the analysis of the spatial distribution of the built-up area is mainly based on a series representing the evolution of Berlin (Fig. 1).

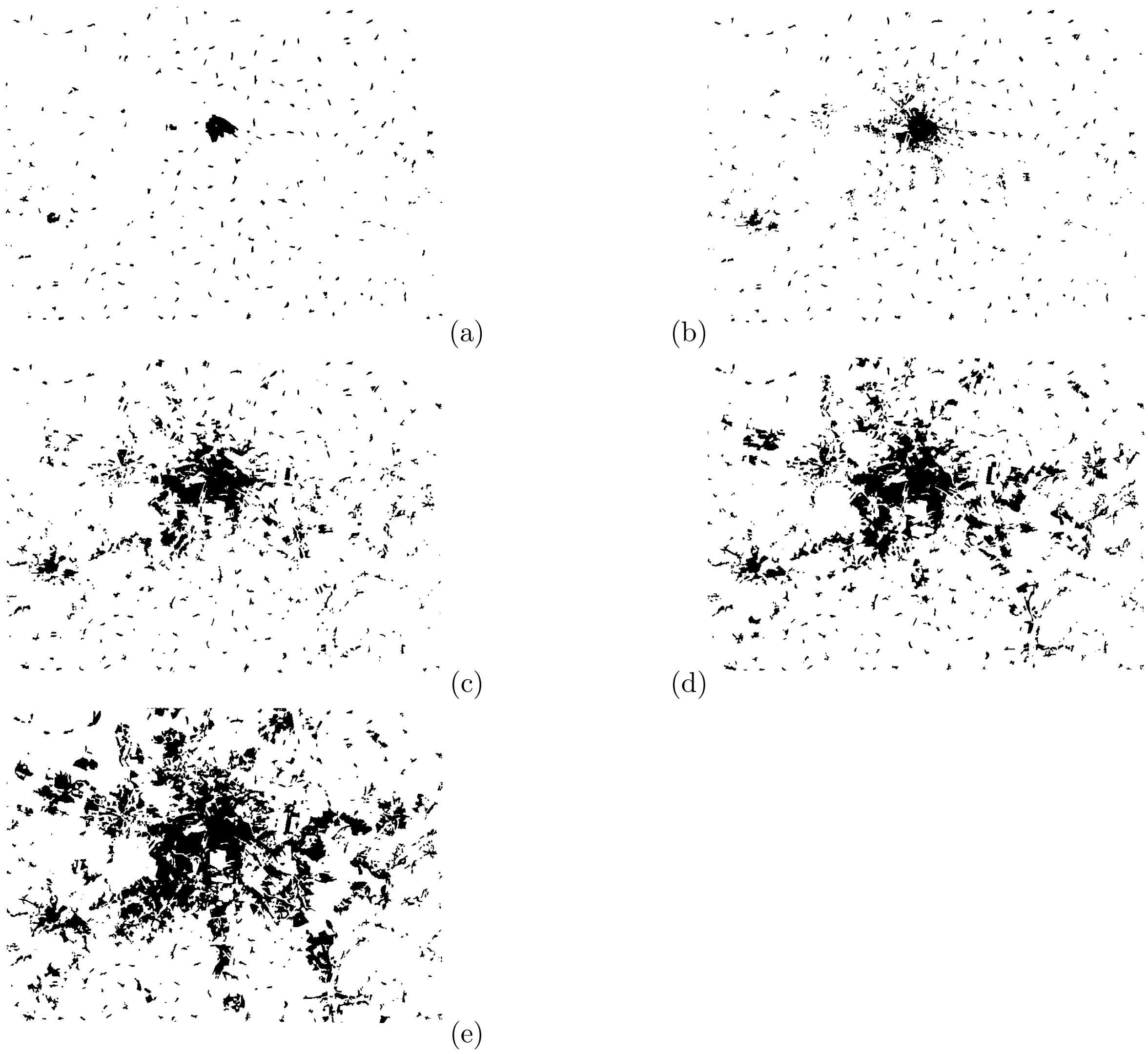


Figure 1: Region of Berlin (a) 1800, (b) 1875, (c) 1910, (d) 1920, (e) 1945 (scale: 1:700.000)

Fig. 1 shows, that the city of Berlin is clearly distinguishable from the surrounding area only by 1800. During its further urban evolution, Berlin seems to merge more and more with the settlements in the neighborhood. The resulting metropolitan area is now denoted as the urban cluster of "Berlin" regardless of the independent cities like Potsdam or Oranienburg which are part of this cluster, too. This leads to different questions, which remain not only for Berlin:

(1) Where is the *center* of the urban cluster? Is the center of gravity of the cluster still identical with the historic town center?

(2) What is the *radial expansion* of the urban cluster? Is it still possible to define a border of the city and to distinguish between the city and the surrounding area?

(3) Due to the distinct dissection of the urban cluster, large empty areas within the urban aggregation exist. What is the *maximum border distance*, which is the maximum distance between an arbitrary mass point within the cluster and the nearest inner or outer area of free space?

From the perspective of physics, the answer to the first question can be found by calculating the *center of gravity* (center of mass) of the urban cluster. After defining a coordinate system for the digitized map of the cluster, a space vector r_i exists for every black pixel ($i = 1, ..., N$) (mass unit $m_i = 1$), and the center of gravity can be calculated as follows:

$$R_m = \frac{1}{N} \sum_{i=1}^{N} r_i \tag{1}$$

Here, the problem arises that the coordinates of the center of gravity are strongly affected by the sector of the map used for the calculations. E.g. in urban agglomerations with large settlement areas in the outer regions, or with distinct asymmetries regarding the build-up area, the center of gravity can be shifted far from the urban center imaginated.

The radial expansion of an urban cluster is not easily defined, due to the frayed structure typical for fractals. However, an estimation can be obtained by calculating the so-called *radius of gyration*, R_g, which is a weighted measure of the distance of the different pixels from the center of gravity, R_m:

$$R_g^2 = \frac{1}{N} \sum_{i=1}^{N} (r_i - R_m)^2 \tag{2}$$

In Fig. 2, the increase of the radius of gyration for the urban area of Berlin is shown. The plot indicates that the radial expansion of Berlin between 1875 and 1945 could be approximated by an exponential growth law.

In order to get an estimate of the *maximum border distance* for the urban cluster of Berlin, the settlement area of Fig. 1(e) is circumscribed consecutively by isochrones measuring 0.25 km in width, proceeding from without towards the center. As can be obtained from Fig. 3, the distance to the nearest inner or outer border is 2 km at maximum.

This relatively small border distance, even for an extended city as Berlin, indicates that the morphology of urban settlements is not only governed by a minimized distance to the center, but also by a minimized distance to the settlement border. On the structural level, these contradicting demands both determine the fractal organization of the settlement.

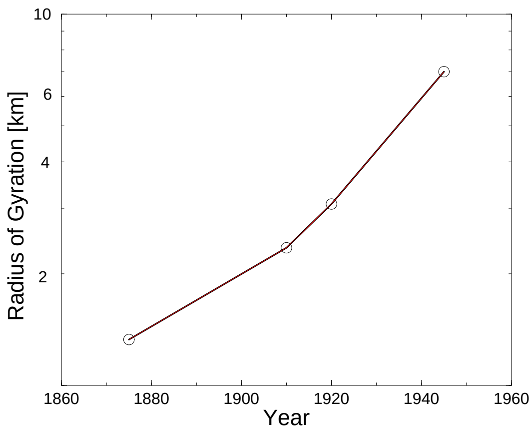


Figure 2: Radius of Gyration in km (logarithmic scale) for the urban cluster of Berlin

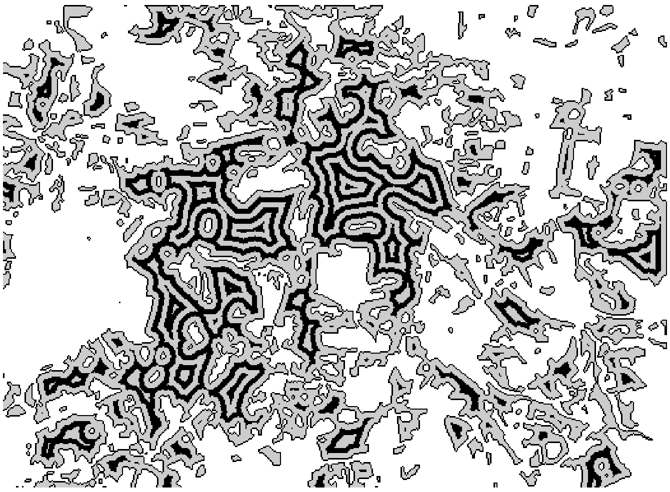


Figure 3: Estimation of the maximum border distance of Berlin (1945) by isochrones measuring 0.25 km in width

2.3 Mass Distribution of the Urban Cluster of Berlin

In order to characterize the settlement area of Berlin in dependence on the distance from the center, around the center of gravity, R_m, concentric circles are drawn with a radius R which is increased by dR for every new circle. Then, $dN = N(R + dR) - N(R)$ is the number of pixels in an ring of area $dA = A(R + dR) - A(R) = 2\pi R dR$. Fig. 4 shows the change of the settlement density, dN/dA in dependence on the distance from the center. The values represent the ratio of the build-up area compared to the total area of the ring, an amount of 1 therefore indicates a complete build-up .

For the years of 1910 and 1920, we find a second ring of very dense settlement appearing in a

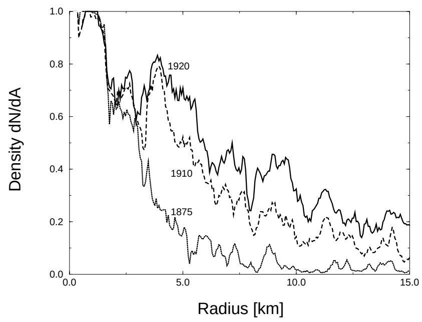


Figure 4: Settlement density (dN/dA) for Berlin vs. distance from the center of gravity in different years: 1875, 1910, 1920

distance of 4-6 km from the center. This indicates the suburbs built for the mass of factory workers during that time. Despite that insight, the plot of Fig. 4 does not seem to be useful for further evaluation. More appropriate is an investigation of the *total settlement area* $N(R)$ in dependence on the distance. Instead of the settlement density, dN/dA, we calculate now $N(R)/A(R)$ which is the total settlement area normalized by the total area, $A(R)$, of the circle (see the double-logarithmic plot in Fig. 5).

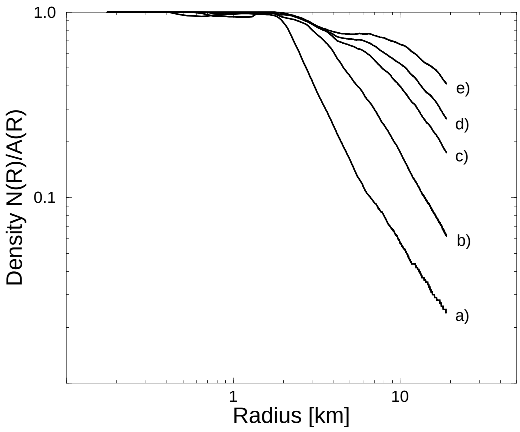


Figure 5: Settlement density $N(R)/A(R)$ for Berlin vs. distance from the center of gravity in a double-logarithmic plot for different years: a: 1800, b: 1875, c: 1910, d: 1920, e: 1945

Fig. 5 shows a compact core of the city (high density) with a radius of about 2 km. Outside this core, the density continuously decreases with a descent which is related to the *fractal dimension* D_f of the settlement area as follows:

$$\frac{N(R)}{4\pi R^2} \sim R^{D_f - 2} \tag{3}$$

D_f is a measure for the descent. As shown in Fig. 5 c-e, some of the regions within the descent part of the plots can be characterized a different descent, and hence by a different fractal dimension. The so-called multi-fractality is a phenomenon typical for inhomogeneous clusters, indicating that, within the urban cluster, more or less compact build-up areas exist.

There are different numerical treatments to calculate the fractal dimension, and the results are slightly different depending on the method used [Frankhauser, 1991, 1994]. In the following, we indicate the method used by an index of the dimension D. The *correlation dimension* D_c should be appropriate to describe the fractal dimension of inhomogeneous fractals. D_c is related to the correlation fuction $c(r)$, which gives a measure whether a certain pixel belongs to the structure:

$$c(r) \sim r^{D_c - 2} \tag{4}$$

$c(r)$ can be calculated by counting the number of pixels in a given ring, as discussed above. However, now the center of the ring is not only the center of gravity, but different pixels are used as centers, and the results are averaged. Fig. 6 gives the correlation function for the urban cluster of Berlin. From Fig. 6, the *fractal dimension of Berlin* is calulated due to eq. (4) as $D_c = 1.79 \pm 0.01$.

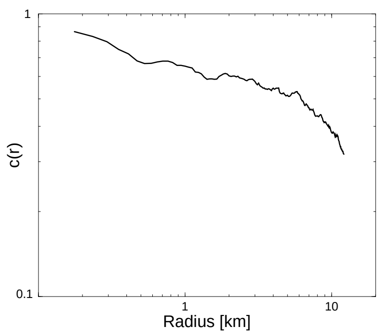


Figure 6: Correlation function $c(r)$ vs. radius in a double-logarithmic plot (Berlin, 1945)

Whereas Figs. 4 and 5 show the build-up density, Fig. 7 shows the build-up area (number of pixels) of Berlin, in dependence on the radius, instead.

In Fig. 7, both the plots for Berlin and for Munich indicate, that the spatial distribution of the settlement area approaches a characteristic power function, $\log A \propto \log R$, in the course of time. This distribution, again, is related to a fractal dimension: $N(R) \sim R^{D_r}$, where D_r now is the radial dimension to indicate the different calculation compared to D_c. From Fig. 7 the following values can be obtained: Berlin (1945): $D_r = 1.897 \pm 0.006$, Munich(1965): $D_r = 1.812 \pm 0.004$. For Berlin, a comparison of both D_r and D_c shows a slightly larger value for D_r, which results from the different definitions.

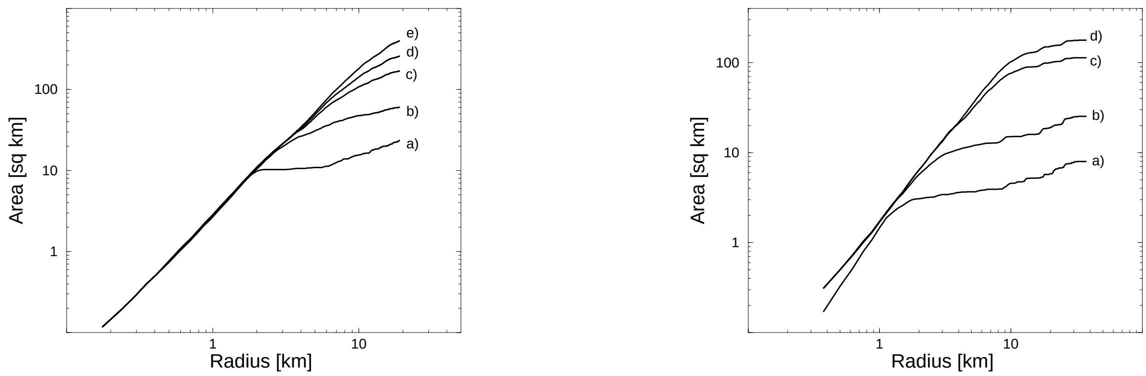


Figure 7: Build-up area $N(R)$ vs. radius in a double-logarithmic plot: (left) Berlin (a: 1800, b: 1875, c: 1910, d: 1920, e: 1945), (right) Munich (a: 1850, b: 1900, c: 1950, d: 1965)

In Fig. 7, the plots for different years are nearly identical for small distances from the center, indicating that almost no additional build-up occured in the central area, but the settlement grew mainly in the outer regions, near the border to the surrounding free space. The value of the radius at which the different plots deviate from the power law, turning into a horizontal line, can give an estimate for an *effective settlement radius* R_{eff}, characterizing the radial expanison of the city. For Berlin, R_{eff} is about 2 km in 1875, but the plots more and more approach the power law in the course of time, which means that the settlement continously grows and merges with the surrounding area. The final value approached, $N(R_{max})/4\pi R_{max}^2$, gives an estimate for the *mean density* of the urban cluster of Berlin, which is $\rho = 0.43$ pixel/lattice unit for the considered map.

3 Analysis and Simulation of the Rank-Size Distribution of Urban Clusters

3.1 Analysis of the Rank-Size Distribution of Urban Clusters

Usually, urban aggregates are not just one merged cluster, but consist of a large number of separated clusters of different sizes. One of the possibilities to describe the composition of the aggregate is the so-called “gradation curve” [Humpert, Brenner, 1992], also discussed in this book.

From the perspective of physics, we can do a cluster analysis which gives the size and the number of the different clusters forming the urban aggregate [Schweitzer, Steinbrink, 1994]. First, we have to determine whether a certain cluster is connected to other clusters or not. Each of the separated clusters gets a number (label), e.g. by using the Hoshen-Kopelman algorithm. Afterwards, the number of pixels of each labeled cluster is determined, and the clusters are sorted with respect to their size, thus determining their rank number (the largest cluster gets rank 1, etc.)

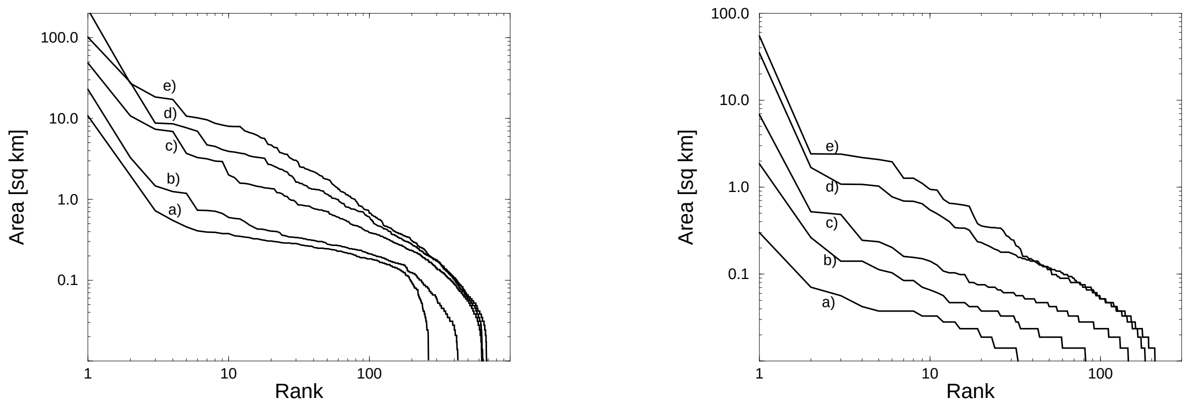


Figure 8: Evolution of the rank-size distribution of connected build-up areas (clusters) in a double-logarithmic plot: (left) Berlin (a: 1800, b: 1875, c: 1910, d: 1920, e: 1945), (right) Munich (a: 1800, b: 1850, c: 1900, d: 1950, e: 1965)

Fig. 8 shows the development of the rank-size cluster distribution for (a) Berlin and (b) Munich.

In particular for the development of Berlin, the rank-size distribution approaches a power law in time, which means with continuous differentiation of the settlement area [Schweitzer, Steinbrink, 1994]. In Fig. 8, this power law is indicated by a straight slope, which can be described by the following equation:

$$\log N(q) = \log N(1) + \alpha \log q \quad \text{or} \quad N(q) = N(1) q^{\alpha}; \quad \alpha < 0 \tag{5}$$

where $N(q)$ is the size of the cluster with rank q and $N(1)$ is the size of the largest cluster which serves as normalization. This power law of the rank-size distribution is also known as PARETO distribution [Frankhauser, 1991, Günther *et al.*, 1992], with α being the PARETO exponent. The PARETO distribution is a characteristic feature for hierarchically organized systems; this means for the considered case that the urban settlement area is formed by clusters of all sizes in the course of time.

If the evolution of urban aggregates occurs that way that the cluster distribution approaches a PARETO distribution in time, we may conclude that the establishment of this distribution could be a measure for a fully developed urban aggregate - and, on the other hand, the deviation from this distribution could be a measure for the *developmental potentiality* with respect to the morphology of the settlement.

Looking at the rank-size distributions of different urban aggregations (cf. Fig. 9), we find that this developed stage is sometimes already reached (cf. Fig. 9 d - Philadelphia)

Table 1 contains the PARETO exponents α which result from Fig. 9 due to eq. (5).

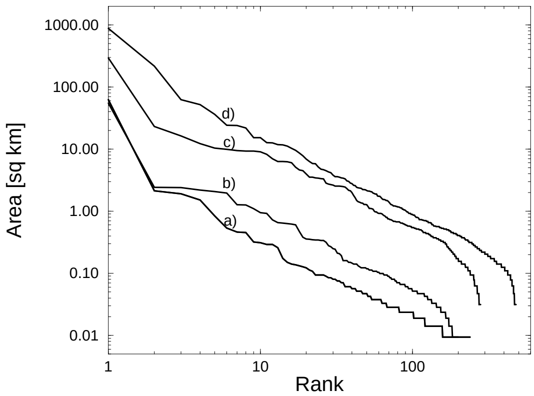


Figure 9: Rank-size distributions of connected build-up areas in a double-logarithmic plot: (a) Daegu (1988), (b) Munich (1965) (c) Moscow (1980), (d) Philadelphia (1980)

City	Pareto Exponent
Munich 1965	- 1.23 ± 0.02
Daegu 1988	- 1.31 ± 0.03
Moscow 1980	- 1.15 ± 0.02
Philadelphia 1980	- 1.32 ± 0.01

Table 1: Pareto exponent α for different urban agglomerations. The values are obtained from Fig. 9 by linear regression of the first 100 ranks.

Our investigations show that the Pareto exponents of different urban agglomerations have similar values. This indicates that the Pareto exponent could serve, in the scope of certain error limits, as an *structural measure of developed urban aggregates*, which may complement the fractal dimension.

In the following, we want to derive a kinetic model to describe the evolution of the rank-size distribution towards a Pareto distribution.

3.2 Simulation of the Rank-Size Distribution of Urban Clusters

3.2.1 The Master Equation

Because urban aggregates consist of clusters of different sizes, we introduce the cluster-size distribution $\mathbf{n}$ of the aggregate as follows: $\mathbf{n} = \{n_1, n_2, ..., n_k, ..., n_A\}$. Here, n_k is the size (the number of pixels) of the cluster with number k, and a total number of A clusters exist ($k = 1, ..., A$). The total number of pixels is obtained by summation:

$$N_{tot}(t) = \sum_{k=1}^{A} n_k \tag{6}$$

and should, for an evolving settlement, increase in the course of time. Fig 10 shows the increase for the urban cluster of Berlin, as obtained from the time series of Fig. 1. As shown, the total increase of the urban area of Berlin follows an exponential growth law, for the years from 1870 to 1945.

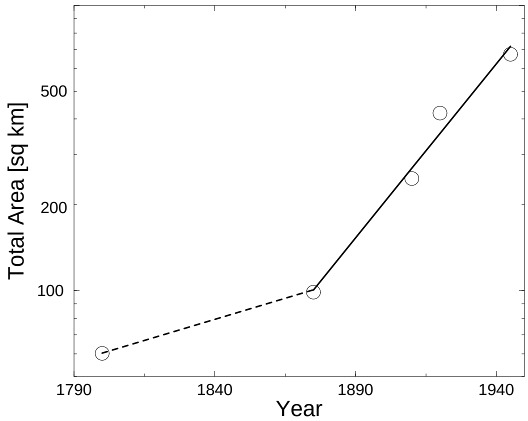


Figure 10: Growth of the settlement area of Berlin in a logarithmic plot

The cluster distribution **n** can be changed due to two fundamental processes: (1) the formation of new clusters, (2) the growth of already existing clusters [Schweitzer, Steinbrink, 1997, 1998]. In principle, we should also consider the shrinkage and disappearance of already existing clusters. However, the probability of these processes is rather small for growing urban areas, and therefore are neglected in the following.

The formation of new clusters can be described by a symbolic reaction $A \xrightarrow{w_1} A+1$, which means that the total number of clusters A increases by one at that time. The probability for forming a new cluster during the next time intervall, w_1, should certainly depend on the total growth of the urban settlement, $N_{tot}(t)$, which can be approximated using empirical data (cf. Fig. 10). Thus, we find the ansatz:

$$w_1 = w(A+1, t+1|A, t) = c(N_{tot}) \tag{7}$$

However, for simplicity we assume a constant probability for the formation of new clusters, i.e. $c = const.$, in the following.

The growth of the already existing clusters can be described by the symbolic reaction $n_k \xrightarrow{w_k} n_k + 1$, which means that the size n of cluster k increases by one at that time. The probability that this process occurs during the next time intervall, w_k, should depend on the already existing size of the cluster and on the existing cluster distribution. Hence, we find the following ansatz:

$$w_k = w(n_k + 1, t + 1 | n_k, t) = \gamma \frac{n_k}{N_{tot}}, \qquad \gamma = 1 - c(N_{tot}) \tag{8}$$

In physical terms, the dependence on $N_{tot}(t)$ represents a *global coupling* between the growth processes of all separated clusters. Therefore, the growth probability of a specific cluster within the urban aggregate is not independent of the other clusters existing.

By means of the factor γ the ratio between the formation of new clusters and the growth of existing clusters can be weighted. Hence, γ depends on the probability of cluster formation, c. If we consider the total increase of the urban area, $N_{tot}(t)$, a known function, then the value of γ determines in the computer simulations, whether the urban growth mainly results from the formation of new clusters, or from the growth of already existing clusters. Usually, we have chosen values of $\gamma = 0.90...0.99$, which means $c = 0.1...0.01$ accordingly.

In a given time intervall, the morphology of the urban aggregate can be changed due to numerous processes of formation and growth of clusters which occur simultanously. Since all possible processes have a certain probability, the result of the evolution is not clear determined. Instead of a deterministic approach we have to use a *probabilistic approach* considering the uncertainity of the future and the limited predictability of evolution.

Therefore, we introduce a probability distribution $P(\mathbf{n}, t) = P(n_1, n_2, ..., n_k, ..., n_A, t)$ which gives the probability that a particular of all possible cluster distributions $\{n_1, n_2, ..., n_k, ..., n_A\}$ will be found after a certain time. The change of this probability distribution in the course of time can be described by a so-called *master equation* which considers all possible processes leading to a change of a particular distribution:

$$\begin{aligned} \frac{\partial P(\mathbf{n}, t)}{\partial t} = & -\sum_i w_k(n_k, t) P(n_1, ..., n_k, ..., t) \\ & + \sum_{i \neq 1} w_k(n_k - 1, t) P(n_1, ..., n_k - 1, ..., t) \\ & - w_1(A, t) P(n_1, ..., n_A, t) \\ & + w_1(A - 1, t) P(n_1, ..., n_{A-1}, t) \end{aligned} \tag{9}$$

In the following section, the master equation will be solved by means of computer simulations. It is rather complicated to derive analytical results for this equation, since all possible processes depend on the others, too, because of the global coupling, given by eq. (6). However, this coupling can be neglected using the assumption, that the total number of clusters, A, is indeed large, but much less then the total number of pixels, N_{tot}. This is basically a restriction for the probability of formation of new clusters, c, which has to be small enough. With the assumption: $1 \ll A \ll N_{tot}$, the master equation can be resolved:

$$P(n_1, ..., n_k, ..., t) \approx \prod_{k=1}^{A} P(n_k, t) \tag{10}$$

Eq. (10) means, that instead of the probability of the whole cluster distribution, $P(\mathbf{n}, t)$, only the probability to find clusters of a certain size, $P(n_k, t)$ has to be considered now. With this probability, the mean size of a cluster k at a given time can be calculated: $\langle n_k(t) \rangle = n_k P(n_k, t)$, and we obtain the result [Günther *et al.*, 1992]:

$$\langle n_k(t) \rangle \sim N_{tot}\, k^{-(1-c)} \tag{11}$$

Eq. (11) indicates an exponential relation between the size of a cluster, n_k, and its number (rank), k, as already discussed for the PARETO-ZIPF distribution, eq. (5). Here, the PARETO exponent α is given by $\alpha = c - 1 < 0$. Thus, we conclude that the change of the cluster distribution, described by the master equation, eq. (9), occurs that way that eventually, in the average, clusters of all sizes exist, which lead to a hierarchical composition of the urban aggregate.

The descent of the PARETO-ZIPF distribution expressed by α depends remarkably on the probability for the formation of new clusters, c. For a large values of c, the distribution becomes flat, for smaller values of c, on the other hand, the distribution becomes more descent, indicating a stronger hierarchical organization, which is mainly determined by the growth of the largest cluster.

3.2.2 Results of Computer Simulations

In order to show a real example for the evolution of the rank-size cluster distribution, we have simulated the development of Berlin for the years between 1910 and 1945 [Schweitzer, Steinbrink, 1997, 1998]. The initial state for the simulation is given by the urban cluster of Berlin (1910) (cf. also Fig. 1 c). The simulations are carried out for two time intervalls: (1) from 1910 to 1920, (2) from 1910 to 1945. For these times, the increase of the total build-up area, $N_{tot}(t)$, is known from the empirical data (cf. also Fig. 10).

The stochastic simulations of the urban growth proceed as follows:

(i) At time t, all possible probabilities for the formation of new clusters and the growth of existing clusters are calculated.

(ii) The probability w for every process is compared with a random number drawn from the intervall $(0, 1)$. Only if w is larger than the random number, the related process is carried out, i.e. either cluster growth: $n_k \rightarrow n_k + \Delta n$, or cluster formation: $\Delta n \rightarrow n_{A+1}$ occur. Here, Δn is the the amount of pixels which is just optically recognized for the given resolution of the map (in the simulation of the growth of Berlin, these are 40 pixels). For the formation of new clusters, a random place is chosen, for the cluster growth, the number of pixels are randomly distributed on the cluster.

(iii) Afterwards, the simulation continues with the next time intervall unless the total increase of the build-up area is reached: $\sum \Delta n = N_{tot}(t_{fin})$

The first simulations which have been carried out that way, however proved that the empirical rank-size distribution for Berlin could not be reproduced. The reason for that is obvious: During the growth of urban clusters, in addition to the global coupling mentioned, also a local coupling exists due to the spatial neighborhood of the clusters. This means, that during the growth, the clusters increase their spatial extension and therefore could merge with neighboring clusters - a phenomenon known from physics as coagulation. In this way, two small clusters could very fast create one larger cluster, which affects the PARETO distribution remarkably. Eventually, if all clusters merge - which is known as percolation - the cluster distribution collapses, because only one large cluster exists.

With respect to the growth probabilities of eq. (8), during the simulations, the largest cluster (rank 1) dominates the whole growth process. During its growth, which is proportional to its size, this cluster also merges with the smaller clusters surrounding it, which leads again to a jump in its size. Eventually, the whole cluster distribution collapses. However, such an evolution was not found empirically. Instead, separate clusters are also found in large urban agglomerations, which leads us to the conclusion that the growth probabilities have to be modified at least for clusters with small ranks.

Our simulations [Schweitzer, Steinbrink, 1997, 1998] have proved that the PARETO distribution for the urban area of Berlin could be reproduced with the additional assumption, that the cluster with rank 1 does not grow automatically proportional to its size, but only due to coagulation with neighboring clusters; whereas cluster with rank 2 or higher grow again proportional to their size, due to eq. (8), and of course due to coagulation.

The results of the computer simulations are shown in Figs. 11 and 12 for two different time intervals. The results clearly indicate that the rank-size cluster distribution of Berlin could be well reproduced for the year of 1920, and even better for the year of 1945, which confirms our kinetic assumptions for the simulation.

The simulations also indicate that the probability for the formation of new clusters, c, should have decreased in the course of time, since the good agreement with the empirical distribution for the year of 1945 was found with a value for c, which is less than half of the one used for the simulation of 1920.

We want to point out again, that from the empirical data only the information about the total increase of the build-up area has been drawn, which could have been also calculated from an exponential growth law. We did not use any information which of the different clusters grew and by which amount - this process has been entirely simulated by the master equation.

The agreement between the simulations and the real rank-size cluster distrubution only demonstrates, that evolution of the size distribution of different clusters has been described correctly. This means an estimation how much a cluster of a particular size should grow during the time interval considered - independent of its definite location within the urban aggregate. These simulations are not able to reproduce the *real* spatial distribution of the different clusters, since no informations

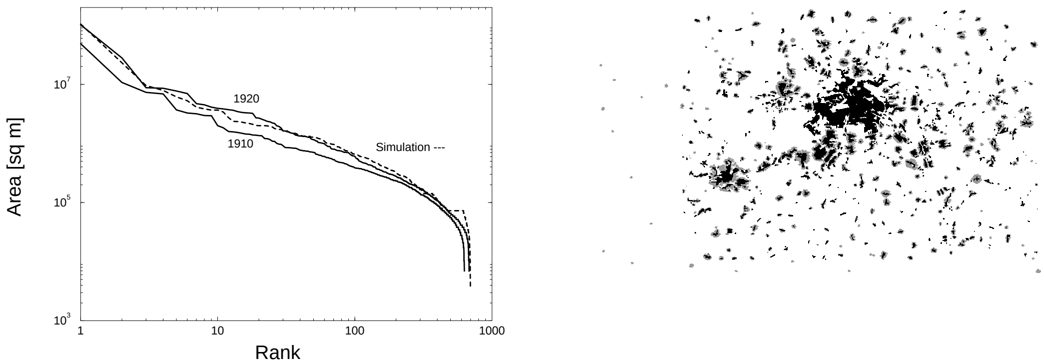


Figure 11: (left) Simulation of the rank-size cluster distribution of Berlin (1910 - 1920), (right) Simulated spatial distribution of the urban clusters for Berlin (1920), black: Initial state (Berlin 1910), grey: simulated growth area. Parameter: c=0.09

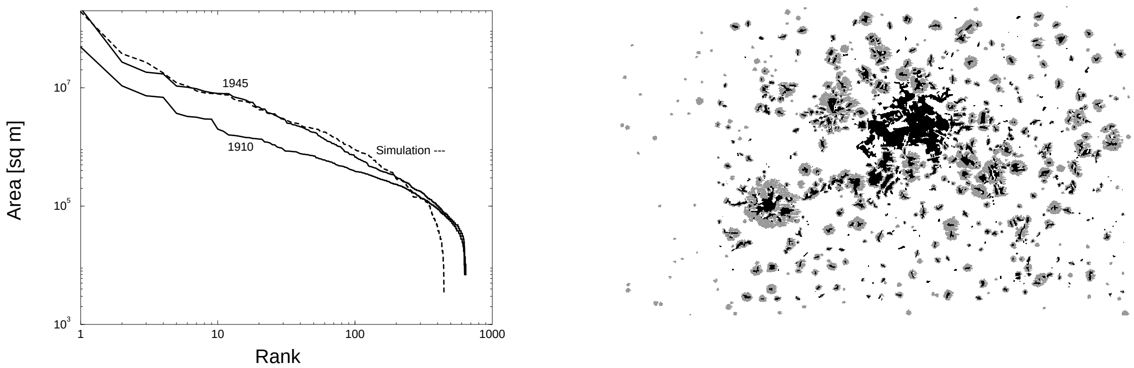


Figure 12: (left) Simulation of the rank-size cluster distribution of Berlin (1910 - 1945), (right) Simulated spatial distribution of the urban clusters for Berlin (1945), black: Initial state (Berlin 1910), grey: simulated growth area. Parameter: c=0.04

about spatial coordinates are considered in the kinetic assumptions. With respect to Berlin, we do not draw conclusions whether "Lichtenrade" or "Zehlendorf" (two specific suburbs) grow, we only give estimates about the development of a cluster of a certain size. However, these estimates sufficiently agree with the real evolution of the urban aggregate, as the simulations have shown.

This gives rise to simulate also the future evolution of urban aggregates which, with respect to their structural morphology, today are similar to an early stage of the evolution of Berlin. As one example, we have simulated the evolution of the South-Korean metropolis Daegu. Despite the fact, that there are living millions of inhabitants today, the distribution of the build-up area of Daegu in 1988 (cf. the black area in the right part of Fig. 14) resembles the situation of Berlin in

1800 or 1875 (cf. Fig. 1). In both cases, one large cluster dominates the whole structure, whereas in the surrounding area only very small clusters exist. Hence, from a structural perspective, the developmental resources, discussed in Sect. 3.1. are rather large.

From a time series, showing the development of the settlement area of Daegu, we know the growth rate in the past. As for the simulation of Berlin, we assume an exponential growth which allows an extrapolation of the future growth. Thus we find an estimate for the increas of the total area, $N_{tot}(t)$, for Daegu (cf. Fig. 13).

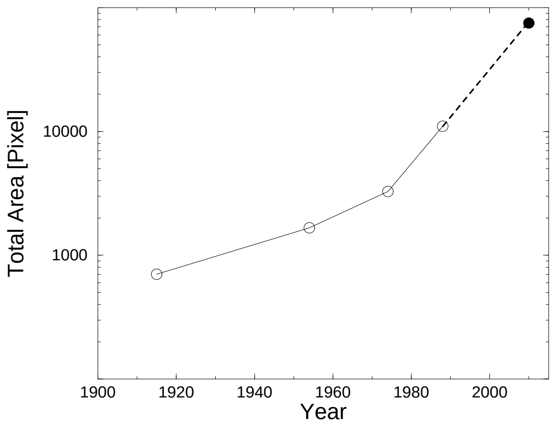
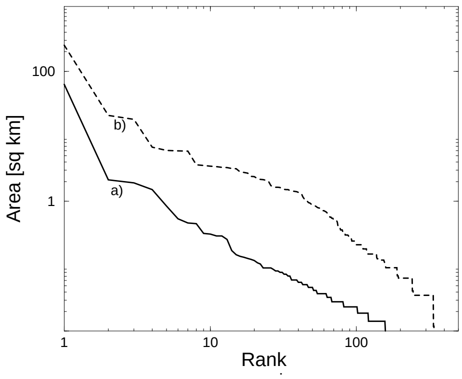


Figure 13: Increase of the build-up area of Daegu in a logarithmic plot. The black dot indicates the estimated growth up to the year 2010 obtained from an exponential growth law.

For the simulations, we have used the assumptions for the formation of new clusters and the growth of existing clusters, as discussed above (eqs. 7, 8). As for the simulation of the growth of Berlin (1910-1945), it is further assumed that the cluster with rank 1 does not grow proportional to its size, but only by coagulation with neighboring clusters. Starting with the known spatial distribution of the settlement area of Daegu in 1988, we have simulated the possible growth of the settlement up to the year of 2010, which is, as in the case of Berlin, a time intervall of about 35 years. The result of the simulations may serve as a first attempt of a prognosis of future development, and is shown in Fig. 14.

The simulations indicate again that, for the future evolution, the rank-size cluster distribution of the urban aggregate may approach the characteristic PARETO distribution. With respect to the spatial distribution, this evolution is accompanied with a forced growth of the rather underdeveloped suburbs in the region, which may now dominate the growth of the central cluster. Eventually, this process results in the hierachical spatial organization typical for a fully developed urban aggregate containing clusters of all sizes.



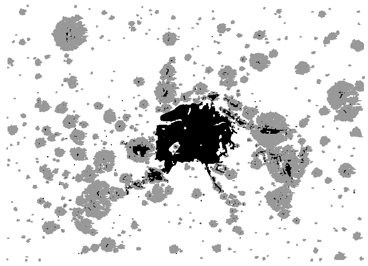


Figure 14: (left) Simulation of the rank-size cluster distribution of Daegu (1988 - 2010), (right) Simulated spatial distribution of the urban clusters for Daegu (2010), black: Initial state (Daegu 1988), grey: simulated growth area. Parameter: c=0.01

4 Outlook: Simulating Urban Aggregates with Reaction-Diffusion Systems

As shown in the previous section for the example of Berlin, the simulations of the rank-size cluster distributions show a good agreement with the empirical distribution, if the additional assumption is used that the largest cluster (rank 1) only grow by coagulation with smaller clusters in the neighborhood. This, however, is an *ad-hoc* assumption, which needs a more detailed discussion.

In fact, this assumption means that, during an early stage of evolution, the central cluster indeed grows proportional to its size, due to eq. (8), but in a later stage, this growth process burns out, wheras smaller clusters near the central cluster further grow and eventually coagulate with it. This may result in an *indirect* growth of the central cluster. Obviously, the direct growth of the central cluster decreases in the course of time.

Thus, our simulations discussed in the previous section, leave us with the important question, *why* does the cluster of rank 1 stop its direct growth, and *when* does it stop this growth.

Similar observations in growth processes are known from different physico-chemical systems. Hence, our idea is to derive a model based on coupled reaction-diffusion equations with different components, which may simulate the urban growth process. In particular, these equations should help in determining the *space-dependence* of the growth probabilities, not discussed so far.

A reaction-diffusion system, suitable for modelling urban growth, is the so-called "activator-inhibitor" model. Here, two different "chemical" components surround the urban cluster, which can diffuse on different time scales. Additionally, "building units" for the urban cluster exist, which can also diffuse and may aggregate on sites suitable for growth. Which of the different sites near the urban cluster should be a growth site, will be determined by the ratio of the activator and the inhibitor at the given time and space coordinates. If the concentration of the activator prevails, the site is "active", meaning it is a possible growth site. On the other hand, if the concentration of the inhibitor prevails, the site is blocked.

From empirical observations of urban growth, two typical scenarios for the late stage of the urban evolution can be obtained:

(i) The slow down of cluster growth until it stops finally. In this case, the urban growth typically *jumps* from the burned out cluster to other urban clusters in a certain distance.

(ii) The continuing cluster growth until most of the cluster have merged with those of the neighborhood. (A typical example is the German Rhine-Ruhr area)

These scenarios could be simulated with activator-inhibitor systems. In our model, e.g. the burn-out of cluster growth and the jump to new growth centers in a certain distance may result from the circumstance, that the concentrations of both the activator and the inhibitor change in time and space, and therefore previous growth sites could be inhibited after a while. This leads to a move of the urban growth zones towards the border of the urban aggregation, or to regions where no inhibitor exists yet. On the other hand, if we have a very fast diffusion of the "building units", a forced spatial aggregation occurs, leading to coalescence of the clusters *before* the inhibitor concentration locally prevails and stops the growth process. In this way, the scenarios outlined above, could be simulated in one model, with critical parameters existing to obtain the crossover between both scenarios.

Based on these ideas, a specific reaction-diffusion system, denoted as *A-B-C model* has been introduced to simulate the spatial growth of urban agglomerations [Schweitzer, Steinbrink 1997]. Here, in addition to the global coupling of urban cluster growth discussed in Sect. 3.2., a local coupling is considered by means of a self-consistent field, generated by the existing urban clusters. This field reflects the attraction of the metropolitan area and influences the local growth probabilities, thus producing a non-linear feedback between the existing clusters and the further urban growth. The microscopic dynamics of this structure formation is based on *active Brownian particles* [Schweitzer, Schimansky-Geier, 1994, Schimansky-Geier *et al.*, 1997], whereas on the macroscopic scale coupled reaction-diffusion equations can be derived.

The model outlined is able to describe several characteristic features of urban growth: **(i)** the formation of new clusters within the urban aggregate which eventually leads to a distribution of separated clusters instead of *one* merged mega-cluster, **(ii)** the formation of depletion zones near the dominating clusters (small ranks), resulting in a local slow down / burn-out of urban growth, **(iii)** accordingly, the spatial move of the growth zones into outer urban regions, **(iv)** the merger of neighboring urban clusters on one hand, the spatial coexistence of separated urban clusters on the other hand, depending on the evolution of the depletion zones and the generated attraction of the larger clusters.

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