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Aggregation Induced by Diffusing and Nondiffusing Media

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1 Introduction

Gathering of individuals is a widespread phenomenon in biology. Reasons are, for instance, to give each other shelter, to reproduce, to explore new regions, to feed or to endure starvation conditions. In the case of myxobacteria (Dworkin and Kaiser 1993) it is known that they glide cooperatively and aggregate under starvation conditions. During gliding they prefer to use paths which were laid down by themselves. When the final aggregation takes place they glide in streams towards developing mounds which later grow to form so-called fruiting bodies. Other examples of collective aggregation are known from larvae, e.g. of the bark beetle *Dendroctonus micans* (Deneubourg et. al 1990) which clumps to feeding groups. Group feeding often improves individual survival, or allows better exploitation of food resources (Tsubaki 1981, Tsubaki and Shiotsu 1982).

In many cases, the aggregation process occurs via exchange of chemical signals between the entities. Chemotaxis is one of the major communication mechanisms, and has been found, for instance, in the aggregation of cells, like human leucocytes (granulocytes) (Gruler and Boisfleury-Chevance 1994; Tranquillo and Alt 1994) and for the slime mold amoebae (Keller and Segel 1970).

In order to discuss the dynamic process of aggregation, we first introduce a discrete model suitable for lattice-based computer simulations, where particles interact by changing the surface they move on, locally. This model can be applied either to the formation of trails or to the formation of aggregates, as found for myxobacteria and insect larvae. The discrete model is approximated by a PDE-system, a so-called chemotaxis system. We discuss critical parameters of the aggregation process, such as initial population density, production rate of the chemotactic substance, and the presence or lack of diffusion of this substance, in dependence on the chemotactic sensitivity.

2 The discrete interacting particle model

Our model is based on particles moving on a two-dimensional plane, which are able to produce and lay down a substance that changes the surface state locally. The particles are sensitive to

this substance, which may change their further movement, thus resulting in a non-linear feedback between the particles and the secreted substance. This model has been applied to interactive structure formation processes, for instance, in Stevens (1992), Schweitzer and Schimansky-Geier (1994), Schweitzer et. al (1996), Schimansky-Geier et. al. (1996), Othmer and Stevens (1996). Here the particles should represent a biological species. We will discuss first, the case of myxobacteria, which produce so-called slime trails where they are located. Each particle sensing a slime trail will prefer to encounter it instead of gliding into an untrailed area. Second we consider the case of larvae, which aggregate to feed. The aggregation occurs due to chemotactic response to a diffusing chemical substance produced by the larvae. This might also be the case for myxobacteria.

Both cases can be discussed within a unified approach. In a discrete model, the movement of the particles is described as a random walk on a two-dimensional lattice, where A^2 denotes the lattice size. For the simulations, two different kinds of lattice are used: (i) square lattice, (ii) triangular lattice, which has the advantage of spatial isotropy. In each case, we assume periodic boundary conditions for the lattice and define N_x to be the set of nearest neighbours of $x \in A^2$.

To keep close to biological motion, the movement of the particles is characterized by the persistence q, indicating that the particles tend to move into the direction of the last step rather than choosing a completely random direction. Let $h_i(t)$ be the orientation of the i^{th} particle at time t, that is, the direction it has been moving in the previous time step. Then the preference for the direction of orientation is described by

$$d(x_0, y, h_i(t)) = \begin{cases} q \geq 1.0\ , & \text{if } y \text{ is neighbour in direction } h_i(t) \text{ of the orien-} \\ & \text{tation of the particle, which is located at } x_0 \\ 1.0\ , & \text{else .} \end{cases}$$

Additionally, the movement of the particles is affected by the substance to which they are sensitive. This can be the slime, as in the case of myxobacteria, or a chemotactic substance, as in the case of larvae and myxobacteria. Let $S : \mathbb{N} \times A^2 \to \mathbb{R}_+$ describe the concentration of the substance in time and space, where $S(0, x) = 1.0$. Then the probability of the i^{th} particle, located at x_0 at time t, to move to point $x \in N_{x_0}$ is

$$P_{i,x_0}(t+1, x) = \frac{S(t,x) \cdot d(x_0, x, h_i(t))}{\sum\limits_{y \in N_{x_0}} S(t,y) \cdot d(x_o, y, h_i(t))} \quad . \tag{1}$$

Let $I_x(t)$ denote the number of particles covering the point $x \in A^2$ at time t. Then $S(t,x)$ can be changed during the next time step by decay via decomposition due to the loss rate λ, by production via $Q(S(t,x), I_x(t))$ and by diffusion, with a diffusion constant D_S. Hence,

$$S(t+1,x) = (1-\lambda)S(t,x) + Q(S(t,x), I_x(t)) - D_S \left(S(t,x) - \tfrac{1}{N_x} \textstyle\sum_{y \in N_x} S(t,y) \right) \ .$$

We assume that the particles produce '$\alpha \times$ the existing concentration of substance' at the point where they are located and additionally a fixed amount β, where $\alpha, \beta \geq 0$. So

$$Q(S(t,x), I_x(t)) = \alpha^{I_x(t)} S(t,x) + \beta \tfrac{\alpha^{I_x(t)} - 1}{\alpha - 1} \ .$$

For $\alpha = 1$ a linear production term results: $Q(S(t,x), I_x(t)) = S(t,x) + \beta I_x(t)$, since $\frac{\alpha^{I_x(t)}-1}{\alpha-1} = I(x)$ if $\alpha \to 1$. When dealing with the model for chemotactic response of larvae we use $\alpha = 1$. For the case of myxobacteria we discuss both, linear ($\alpha = 1$) and superlinear ($\alpha > 1$) production of the slime, similar to Davis (1990), who investigated a one-dimensional reinforced random walk without diffusion of the substance. He found that a particle localizes at a random place if the substance is produced superlinearly and does not localize if it is produced only linearly.

Our model describes a reinforced random walk which may result in an aggregation of the particles, due to the non-linear coupling between the concentration of substance and the movement of the particles. A more realistic model of myxobacterial aggregation is given in Stevens (1992), where both slime trail following and the response to a diffusing chemoattractant are needed to get stable centres of aggregation. Here, we restrict ourselves to the simpler model of only one substance, in order to get a better control of the parameters and to check their relevance.

3 Computer simulations

In our model of interacting particles, there is an interplay between the parameters describing the performance of the particles themselves, like the persistence, q, or the production of substance, α, β, and the parameters which describe the evolution of the chemical substance, as decay or diffusion, λ, D_S.

In our computer simulations we consider a square lattice A^2 of 70×70 gridpoints with periodic boundary conditions. This gridsize is chosen to avoid too strong boundary effects and at the same time guarantee a clear output. Initially, $N = 1000$ particles are randomly distributed on the inner square lattice A_1^2 of 30×30 gridpoints (see Fig. 1a). Each time step, the particles move to one of their four nearest neighbours and interact with the surface, as described above.

No diffusion, no decay, linear production, no persistence: For $D_S = 0.0, \lambda = 0.0, \alpha = 1.0, \beta = 10^{-5}$, with $S(0,x) = 10^{-4}$, and $q = 1.0$, the simulations show swarming of the particles if the chemical substance is laid down and measured inbetween the gridpoints (see Fig. 1b, 200 time steps), as described by Davis (1990) for a single particle. If the substance is laid down and measured directly on the gridpoints as described in the model equations a stronger taxis effect results (compare Fig. 2a and Othmer and Stevens (1996)). If the initial particle density, N/A_1^2, exceeds a critical value, the swarm remains more local, and the particles form small clusters.

Diffusion, no decay, linear production, no persistence: Fewer clusters appear if the substance diffuses with $D_S = 0.05$, but more particles are trapped, compared to the situation without diffusion (see Fig. 2b, 1000 time steps). In this case the initial conditions play a less important role, since diffusion of the attractive information effaces initial clusters and guides particles from regions with only few particles, towards regions with developing aggregation centres. For increasing D_S, the aggregation centres interfere even more than shown in Figure 2b, and no separated clusters are formed. Hence, in this case, the existing clusters can only be stabilized when the production of substance is increased and a higher initial concentration of the particles is given.

Figure 1: a) Initial conditions. b) Reinforced random walk of 1000 particles, where the substance is laid down and measured inbetween the grid points. Grey dots mark the paths the particles have used. Black squares mark a single particle, the squares of different grey levels mark 2 to 9 particles and white squares mark 10 and more particles.

Figure 2: a) Reinforced random walk of 1000 particles, where a non-diffusing substance is laid down on the grid points. b) Same situation as in a) but with diffusion of the substance.

No diffusion, no decay, superlinear production, no persistence: For a superlinear production ($\alpha = 1.01$) and no diffusion of the chemical substance, well separated aggregates are formed very quickly (see Fig. 3a, 200 time steps). This effect is amplified if the particles have a high initial density. However, if the particles are more distant initially, they are trapped in many very small clusters.

Diffusion, no decay, superlinear production, no persistence: Adding diffusion ($D_S = 0.05$), the streaming towards the clusters becomes stronger, the clusters get bigger and become well separated (see Fig. 3b).

Decay of the chemical substance amplifies the effect of aggregation; however particles far away from the centres have a tendency to jump back and forth between two gridpoints. This effect will be

Figure 3: (a) Superlinear production of the substance. (b) Same situation with diffusion of the substance.

smoothed out by increasing D_S, which results, again, in streaming towards the clusters.

Diffusion, decay, superlinear production, persistence: To keep close to the myxobacterial behaviour, the simulations are now carried out with persistence, $q = 3.0$. The particles swarm out and return to the aggregates more easily, once they have chosen the correct direction. This behaviour is close to reality, and can be supported by choosing $D_S \neq 0$ (Fig. 4 shows 5000 particles after 200 time steps). Here diffusion, $D_S = 0.05$, and decay, $\lambda = 0.05$, stabilize the aggregation centres. In the more complex model for myxobacterial aggregation, described in Stevens (1992), superlinear slime production does not affect aggregation in the way discussed here. Further research will be done to understand this.

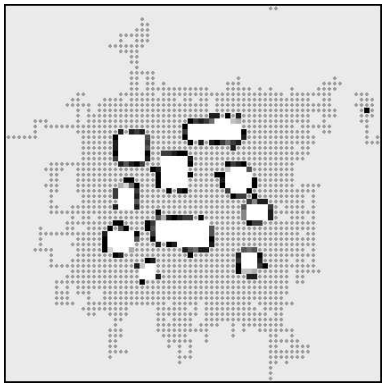

Figure 4: Persistence of the particles and decay of the substance.

In our simple model a fine tuning of the parameters accounts for swarming of the particles, aggregation and stabilization of the aggregation centres. A further explanation of the computer simulations is given in the following section based on a continuity approximation.

4 Continuity approximations for the discrete particle model

The interacting particle model introduced in the previous section can be approximated by a continuous model. First a diffusion approximation can be carried out for the simplest version of the four nearest neighbour reinforced random walk of *one* particle in two dimensions, that is equation (1) , where $q = 1.0$. This results in the following chemotaxis-equation for the probability of the particle to be located at point x at time t (Othmer and Stevens 1996):

$$\partial_t p \quad = \quad D \nabla \left(\nabla p - \chi \frac{p}{s} \nabla s \right) \ . \tag{2}$$

Here $\chi = 2$ and $s(t,x)$ denotes the density of the chemical substance, which satisfies the reaction diffusion equation $\partial_t s = D_s \Delta s + b(s,p) - \lambda s$, where D_s and $\lambda \geq 0$, and b is a suitable functional for the growth of the chemical substance.

In the following we consider $b(s,p) = \beta \cdot p$. For $D_s = 0, \lambda = 0$ and with an initial peak for $p(0,x)$, only a high production rate of s accounts for blowup of p in finite time. For a low production rate of s an initial peak of p breaks down. This is closely related to Davis' (1990) results on the reinforced random walk of a single particle, where the approximation yields equation (2) with $\chi = 1$. Then blowup for p occurs only for superlinear growth of s, which reflects the localization result. If the decision of the particles is gradient based, i.e. the transition rates equal $a_1 + a_2(s(t,x) - s(t,x'))$, where $x' \in N_x$ and $a_1, a_2 \in \mathbb{R}$, one obtains

$$\partial_t p = D' \nabla \left(a_1 \nabla p - 2\, a_2 p \nabla s \right) \quad , \quad \text{respectively} \quad \partial_t p = D \Delta p - \nabla p \left(\chi \nabla s \right) \ , \tag{3}$$

and $D = D'\, a_1$ and $\chi = 2 D a_2 / a_1$ (Othmer and Stevens 1996).

Now (3) can be derived as a limiting equation for the behaviour of *many* interacting particles. In our model the chemical substance determines the motion of the particles. They search for local maxima of $s(t,x)$, so the dynamics of the i^{th} Brownian particle in the N-particle system is described by the following Langevin equations:

$$\frac{dx_i}{dt} = v_i \qquad ; \qquad \frac{dv_i}{dt} = -\gamma v_i + \nabla s(t,x_i) + \sqrt{2\,\epsilon\,\gamma}\ \xi_i(t) \ , \tag{4}$$

where γ is the friction coefficient and $\xi_i(t)$ is Gaussian white noise with intensity ϵ. The probability of finding N particles in the vicinity of $x_1, ..., x_N$ on a surface A at time t can be formulated in terms of the canonical N-particle distribution function $P(t, x_1, ..., x_N)$. In the limit of strong damping, $\gamma \to \infty$, it reads:

$$\frac{\partial}{\partial t} P(t, x_1, ..., x_N) \quad = \quad - \sum_{i=1}^{N} \left[\nabla_i \left(\chi\, \nabla_i\, s \right) P - D\, \Delta_i P \right] \ . \tag{5}$$

Here $\chi = 1/\gamma$ denotes the mobility of the particles and $D = \epsilon/\gamma$ the spatial diffusion coefficient for the density of the Brownian particles. In the mean field limit, we obtain the particle density $p(t,x)$ from:

$$p(t,x) = \int dx_1 \ldots dx_{N-1}\, P(t, x_1, ..., x_{N-1}, x) \ . \tag{6}$$

Finally (3) results by integrating (5) due to (6).

¿From a bifurcation analysis of the mean field equation one finds the condition for the instability of the homogeneous state p_0, s_0: $\beta p_0 / \gamma > \epsilon (\lambda + \kappa^2 D_s)$,
where $p_0 = N/A$, $s_0 = p_0 \beta / \lambda$, A is the surface area, and κ the wave number of a fluctuation. If β is large, or if ϵ is small, the particles quickly form several clusters, which result in a local growth of $s(t,x)$, as shown in the simulations below.

Initially the particles are distributed on a triangular lattice and s is produced linearly. First the spikes of the chemical substance grow independently as presented in Figure 5. If the production of the substance becomes stationary, i.e. the decay compensates for the production, a transition into a second regime occurs. Here, the different spikes, which have bound the Brownian particles, compete, leading to a decrease in the number of spikes (see Fig. 6).

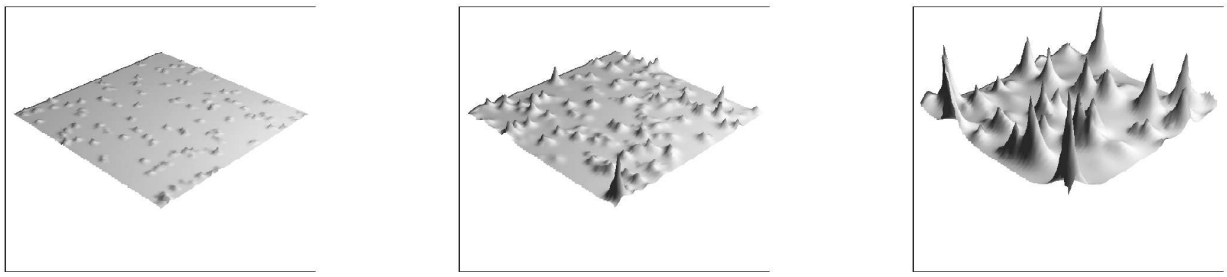

Figure 5: Evolution of $s(t,x)$ generated by $N = 100$ particles during the growth regime. Time in simulation steps: a) $t = 10$, b) $t = 100$, c) $t = 1000$ (lattice size: $A = 100 \times 100$).

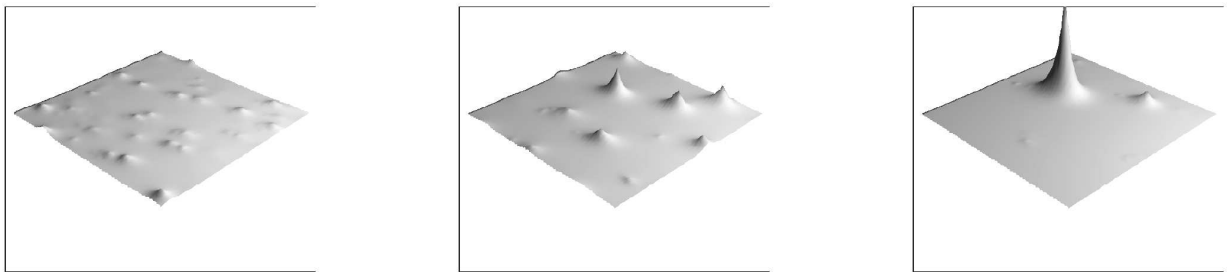

Figure 6: Evolution of $s(t,x)$ generated by $N = 100$ particles during the competition regime. Time in simulation steps: a) $t = 1000$, b) $t = 5000$, c) $t = 50000$. The density scale is 0.1 $\times$ scale of Fig. 5. Hence, the left part of Fig. 6 is the same as the right part of Fig. 5.

The selection among the different spikes can be described in terms of a selection equation of the EIGEN-FISHER type (Schweitzer and Schimansky-Geier 1994), which is

$$\begin{aligned}\partial_t s(t,x) &= \frac{\gamma\ s_0}{\langle \exp[(\chi/\epsilon)\, s(t,x)]\rangle_A} s(t,x) \\ &\quad \times \{\frac{\exp[(\chi/\epsilon)\, s(t,x)]}{s(t,x)} - \frac{\langle \exp[(\chi/\epsilon)\, s(t,x)]\rangle_A}{s_0}\} + D_s \Delta s(t,x)\ ,\end{aligned}$$

where $\langle \exp[(\chi/\epsilon)\, s(t,x)]\rangle_A = \frac{1}{A}\int_A \exp[(\chi/\epsilon)\, s(t,x')]\ dx'$ is similar to the mean "fitness", representing the global selection pressure. Further growth of a spike occurs only as long as $\exp[(\chi/\epsilon)\, s(t,x)]\ s_0 > \langle \exp[(\chi/\epsilon)\, s(t,x)]\rangle_A\ s(t,x)$ holds. Otherwise, the spike will decay again due to the competition process. Provided a suitable neighbourhood, eventually the largest spike will survive, as indicated by the simulations.

A rigourous approach to derive density equations from an interacting many particle system can be found in Stevens (1992). The position of the i^{th} particle $x_N^i(t)$ in an N-particle system is given by $\frac{dx_N^i(t)}{dt} = \chi \nabla s_N(t, x_N^i(t)) + \sqrt{2D}\ \xi_i(t)$. Here $s_N(t, x_N^i(t))$ describes the amount of the chemical substance not only at the point $x_N^i(t)$ but also in its neighbourhood, due to a weight depending on N. Hence the interaction is not local. It is chosen to be moderate, which means that the main interaction range of each individual particle shrinks for $N \to \infty$, but the number of other particles in this range tends to ∞. Under these conditions the many particle system and the continuous model are a good approximation of each other.

5 Other results on the limiting equations

Several results are known about chemotaxis-equations. The stability analysis for a quite general situation was done by Schaaf (1985). Blowup results were given by Jäger and Luckhaus (1992), Nagai (1996), Herrero and Velasquez (1995, 1996a, 1996b) and Biler (1995). The qualitative behaviour in generally nonsmooth domains was considered by Gajewski and Zacharias (1996). In all cases the substance diffuses; sometimes with $D_s \gg 1$. For chemotaxis-equations with $D_s = 0$ qualitative results were given by Rascle and Ziti (1995) (blowup), Othmer and Stevens (1996) and Levine and Sleeman (1996). In the last two papers blowup of p in finite time, finite stable peaks and collapse of developing peaks are disussed.

6 Discussion

We have simulated the aggregation of interacting particles due to a substance produced by themselves. Different parameters, have been discussed with respect to their effect onto aggregation. We note that a large production rate of the slime, combined with a supercritical initial concentration of the particles, results in the formation of aggregation centres, but, on the other hand, prevents swarming. Diffusion of the slime enforces the aggregation effect, but effaces the centres a little bit.

Decay of the slime stabilizes the aggregation centres, but increases the chance that single particles are trapped in certain regions. The persistence of particle movement does not change this qualitative behaviour, however, it makes the simulations more realistic. Further research should be done to compare the qualitative behaviour of the interacting particle model and its continuous approximation. From a numerical point of view, it would be interesting to use the particle model with moderate interaction to simulate the chemotaxis system.

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