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Active Motion in Systems with Energy Supply

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Abstract

Biological motion, human traffic and many other types of active motion rely on the supply of energy. In order to derive a rather general approach for active motion, we have proposed a model of active Brownian particles, which have the ability to take up energy from their environment, to store it in an internal energy depot and to convert internal energy to perform different activities, such as metabolism, acceleration of motion, changes of the environment, or signal-response behavior. The basic description is given by an extended Langevin equation for the motion of the particles, which is coupled to a balance equation for the internal energy depot. Different to the case of passive Brownian motion, we find several new features of the dynamics, such as non-equilibrium velocity distributions, uphill motion, transitions between Brownian and directed motion, or excited collective motion and spontaneous rotations in an ensemble of active particles.

1 Introduction

It is well known from the thermodynamics of irreversible processes that systems may exhibit a rich variety of complex behavior if there is a supercritical influx of free energy. This energy may be provided in different forms, i.e. matter (chemical components, resources), high temperature radiation, or signals. What kind of complex behavior is observed in a system, will of course not only depend on the influx of energy but also on the interaction of the entities that comprise the system. Among the prominent examples that can be observed are processes of pattern formation, and different types of collective motion, such as swarming.

In this paper, we will especially focus on *active motion*, which is a phenomenon found in a wide range of systems. Already in physico-chemical systems a *self-driven motion* of particles can be observed (Mikhailov and Meinköhn, 1997). On the biological level, *active*, self-driven motion can be found on different scales, ranging from cells (Tranquillo and Lauffenburger, 1987; Gruler and Boisfleury-Chevance, 1994) or simple microorganisms up to higher organisms, such as bird or fish (Okubo, 1986, Alt and Hoffmann, 1990). Last, but not least, also human movement can be described as active motion (Helbing *et al.*, 1997), as well as the motion of cars (Helbing, 1997; Helbing and Huberman, 1998). All these types of active motion occur under energy consumption and energy conversion and may also involve processes of energy storage.

In order to derive a general concept of active motion, we need to consider different aspects:

- deterministic influences, such as a preferred or intended direction of motion,
- stochastic influences, which may interfere with the deterministic motion and would be able to change the direction of motion, for instance,
- the take-up of energy needed for active motion,
- the (internal) storage of energy, for instance in a depot, a tank, a battery etc.
- the conversion of stored energy into (kinetic) energy of motion

Our realization of the concept outlined above is based (i) on the Langevin equation which describes the motion of an entity and (ii) on the idea of an internal energy depot, where the taken-up energy can be stored and used for different activities such as acceleration of motion. This model will be described in detail in Sect. 2.

The Langevin approach is well known in physics as a description of Brownian motion. This is the motion of small, but larger than molecular particles which are immersed in a liquid. The British botanist Robert Brown, who in 1827 discovered the random motion of these particles, reported that their motion resembles that of microscopic living creatures. However, as the pioneering work of Einstein, Smoluchowski, Langevin at the beginning of the 20th century have shown, this motion can be completely understood and described by means of physical principles. It means that the motion of a "simple" Brownian particle is due to fluctuations of the surrounding medium, i.e. the result of random impacts of the molecules or atoms of the liquid or gas, the particle is immersed in.

This type of motion would be rather considered as *passive motion*, simply because the Brownian particle does not play an active part in this motion. It is an *undirected* motion, driven by thermal noise. Passive motion can be also directed, e.g. if it is driven by convection, currents or by external fields.

Active motion, on the other hand relies on the supply of energy. Therefore, we have introduced *active Brownian particles* (Steuernagel *et al.*, 1994; Schimansky-Geier *et al.*, 1995, 1997; Schweitzer, 1997; Ebeling *et al.*, 1999), which are Brownian particles with the ability to take up energy from the environment, to store it in an internal depot and to convert internal energy to perform different activities, such as metabolism, acceleration of motion, changes of the environment, or signal-response behavior (Schweitzer and Schimansky-Geier, 1994; Lam, 1995; Schweitzer *et al.*, 1997). The internal energy depot adds a new degree of freedom to the particle, it would allow the particle to gradually spend the energy on its activity and therefore to "overcome" periods where an energy take-up is not possible.

While such an extension of the concept of Brownian particles is evidently inspired by some biological analogies, we want to note explicitely, that we do *not* intend to model a particular system. Our aim is merely to extend a known dynamics by some plausible arguments, thus adding some more complexity to a physical system. We will, on the other hand, show that even these rather simple additional assumptions will lead to remarkable dynamic phenomena, which find an interesting analogy in a variety of systems.

The paper is organized as follows: In the following section, we will give an outline of the basic model both in terms of a particle-based description and in terms of the corresponding density description. In Sect. 3, we will discuss different variants of the one-dimensional model, which consider external and interaction potentials for the particle's motion. In Sect. 4, we investigate the motion active particles and "swarms" in a two-dimensional external potential and investigate the influence of spatially localized energy sources. The final Sect. 5 will discuss related work on active motion done by different authors and will also provide some hints for further applications of the model.

2 Model of driven Brownian dynamics

2.1 Equations of motion and energy balance

The motion of a Brownian particle with mass m, position $\boldsymbol{r}$, and velocity $\boldsymbol{v}$ moving in a space-dependent potential, $U(\boldsymbol{r})$, can be described by the following Langevin equation:

$$\dot{\boldsymbol{r}} = \boldsymbol{v}\,;\;\; \dot{\boldsymbol{v}} = -\gamma_0\,\boldsymbol{v} - \frac{1}{m}\boldsymbol{\nabla}U(\boldsymbol{r}) + \mathcal{F}(t) \tag{1}$$

Here, γ_0 is the friction coefficient and $\mathcal{F}(t)$ is a stochastic force with strength D and a δ-correlated time dependence

$$\langle\mathcal{F}(t)\rangle = 0\,;\;\; \big\langle\mathcal{F}(t)\mathcal{F}(t')\big\rangle = 2D\,\delta(t-t') \tag{2}$$

In the case of thermal equilibrium systems we may assume that the fluctuation-dissipation theorem (Einstein relation) applies:

$$D = \gamma_0\frac{k_BT}{m} \tag{3}$$

T is the temperature and k_B is the Boltzmann constant.

The Langevin equation describes the motion of a Brownian particle within a particle-based approach. In statistical physics, there is a correspondence to the density approach which is based on the distribution function $P(\boldsymbol{r},\boldsymbol{v},t)$. It describes the probability density to find the particle at location $\boldsymbol{r}$ with velocity $\boldsymbol{v}$ at time t. As well known, the distribution function $P(\boldsymbol{r},\boldsymbol{v},t)$ which corresponds to the Langevin eq. (1), can be described by a Fokker-Planck equation of the form:

$$\begin{aligned} \frac{\partial P(\boldsymbol{r},\boldsymbol{v},t)}{\partial t} &= \frac{\partial}{\partial \boldsymbol{v}} \left\{ \gamma_0\, \boldsymbol{v}\, P(\boldsymbol{r},\boldsymbol{v},t) + D\, \frac{\partial P(\boldsymbol{r},\boldsymbol{v},t)}{\partial \boldsymbol{v}} \right\} \\ & \quad -\boldsymbol{v}\, \frac{\partial P(\boldsymbol{r},\boldsymbol{v},t)}{\partial \boldsymbol{r}} + \frac{1}{m} \boldsymbol{\nabla} U(\boldsymbol{r})\, \frac{\partial P(\boldsymbol{r},\boldsymbol{v},t)}{\partial \boldsymbol{v}} \end{aligned} \tag{4}$$

The stationary solution of eq. (4), $P^0(\boldsymbol{r},\boldsymbol{v})$, is known to be the Boltzmann distribution:

$$\begin{aligned} P^0(\boldsymbol{r},\boldsymbol{v}) &= C \exp\left\{ -\frac{U(\boldsymbol{r})}{k_B T} \right\} \exp\left\{ -\frac{\gamma_0}{2\,D}\, v^2 \right\} \\ &= C \exp\left\{ -\frac{1}{k_B T} \left[\frac{m}{2}\, v^2 + U(\boldsymbol{r}) \right] \right\} \end{aligned} \tag{5}$$

where the constant C results from the normalization condition.

This known picture will become more complex if we add a new degree of freedom to the model by considering that Brownian particles can be also pumped with energy. This extension leads to the model of *active Brownian particles*. It is based on the idea that the particles have the ability to take up energy from the environment (Steuernagel *et al.*, 1994; Schweitzer *et al.*, 1998; Ebeling *et al.*, 1999), to store it in an internal energy depot, $e(t)$, and to convert internal energy to perform different activities, such as metabolism, motion, change of the environment, or signal-response behavior.

If we assume that the (external) space-dependent potential $U(r)$ describes the environment, then changes caused by the active particles may result in an additional contribution, which can be summarized in a "effective" or *quasi-potential*:

$$U^{\star}(\boldsymbol{r},t) = U(\boldsymbol{r}) - h(\boldsymbol{r},t) \tag{6}$$

The space and time dependent contribution of the active particles, $h(\boldsymbol{r},t)$, is also denoted as a self-consistent field (Schimansky-Geier *et al.*, 1995, 1997, Schweitzer, 1997). This can be for instance a chemical field used for the communication among the particles (Calenbuhr and Deneubourg, 1991), as it is widely observed in the aggregation of larvae and other biological species (Deneubourg *et al.*, 1990, Stevens and Schweitzer, 1997). Since the Langevin eq. (1) considers that particles respond to the gradient of the potential, the minus sign in eq. (6) indicates that the particles try to follow the *ascent* of the field $h(\boldsymbol{r},t)$, while they also follow the *descent* of the external potential $U(\boldsymbol{r})$.

Taking into account the different activities, the internal depot may be changed due to the following processes:

1. gain of energy from the environment, where $q(\boldsymbol{r}, t)$ is the (space and time dependent) flux of energy into the depot. It may be a random function because of the stochastic variable $\boldsymbol{r}$.
2. loss of energy due to internal dissipatve processes. The rate of dissipation, c, is assumed to be proportional to the content of the depot.
3. conversion of internal energy for environmental changes, i.e. generation of a self-consistent field $h(\boldsymbol{r}, t)$ at a rate s.
4. conversion of internal energy into kinetic energy with a rate $d(\boldsymbol{v}^2)$ and an efficiency $\eta(v^2)$, which both may depend on the square of the actual velocity, $\boldsymbol{v}$, of the particle. A simple ansatz for the conversion rate $d(v^2)$ reads: $d(v^2) = d_2 v^2$, with $d_2 > 0$.

The resulting balance equation for the energy depot is then given by:

$$\frac{d}{dt} e(t) = q(\boldsymbol{r}) - c\, e(t) - s - \eta(v^2)\, d_2 v^2\, e(t) \tag{7}$$

Considering both the existence of a self-consistent field, eq. (6), and the additional acceleration of the active particle due to conversion of internal into kinetic energy, the Langevin eq. (1) now has to consider additional driving forces and needs to be modified to:

$$\dot{\boldsymbol{v}} = -\gamma_0\, \boldsymbol{v} - \frac{1}{m} \boldsymbol{\nabla} \Big[U(\boldsymbol{r}) - h(\boldsymbol{r}, t) \Big] + \eta(v^2)\, d_2 e(t)\, \boldsymbol{v} + \sqrt{2\,D}\, \boldsymbol{\xi}(t) \tag{8}$$

Here the random function $\boldsymbol{\xi}(t)$ is assumed to be Gaussian white noise. In order to demonstrate the effect of the internal energy depot on the motion of the active particle, we will discuss some special cases in the following sections. Let us from now on set the mass of the particle to $m = 1$, and the flux of energy into the internal depot of the particle as constant: $q(\boldsymbol{r}) = q_0$.

2.2 Non-linear friction functions and free motion

In this section, we restrict the discussion to a constant potential $U^{\star}(\boldsymbol{r}, t) = \text{const}$. That means, environmental changes such as the generation of a self-consistent field and responses to external gradients are neglected here. Let us further assume that the efficiency of conversion of internal into kinetic energy, $\eta(v^2)$, decreases with increasing velocity, for example (Tilch *et al.*, 1999):

$$\eta(v^2) = \frac{\eta_1}{1 + \eta_2 v^2} \tag{9}$$

where η_1 and η_2 are constants. Then, the Langevin eq. (8) reads:

$$\dot{\boldsymbol{v}} = -\gamma_{\text{eff}} \boldsymbol{v} + \sqrt{2\,D}\, \boldsymbol{\xi}(t) \tag{10}$$

where two dissipative terms are combined in an *effective* friction function:

$$\gamma_{\rm eff} = \gamma_0 - \frac{\eta_1\, d_2\, e(t)}{1+\eta_2 \boldsymbol{v}^2} \tag{11}$$

The content of the energy depot $e(t)$ which is in general time-dependent, may be eliminated in two limiting cases:

1. The depot energy is so large, that it may be considered as constant, i.e. $e(t) = \text{const.}$
2. The internal energy is an adiabatic variable which relaxes fast compared to changes of the velocity and space variables, i.e. $\dot{e}(t) = 0$.

In both cases the energy depot may be considered as quasistationary and reads then:

$$e(t) \;\rightarrow\; e_0 = \frac{q_0}{c + d_2 \boldsymbol{v}^2} \tag{12}$$

In this case, the effective friction function becomes a velocity dependent function:

$$\gamma_{\rm eff} \;\rightarrow\; \gamma(\boldsymbol{v}) = \gamma_0 - \frac{\eta_0}{1+\eta_2 \boldsymbol{v}^2} \tag{13}$$

with $\eta_0 = \eta_1 d_2\, e_0$. Dependent on the parameters, the friction function, eq. (13) may have a zero at a definite velocity, $\boldsymbol{v}_0$. Defining the constants η_1, η_2 in relation to the parameters influencing the energy depot (Erdmann *et al.*, 2000),

$$\eta_0 = \eta_1\, d_2\, e_0 = q_0\, \frac{d_2}{c}\;;\; \eta_2 = \frac{d_2}{c} \tag{14}$$

we find for the velocity $\boldsymbol{v}_0$:

$$\boldsymbol{v}_0^2 = \frac{1}{\eta_2}\left(\frac{\eta_0}{\gamma_0} - 1\right) = \frac{q_0}{\gamma_0} - \frac{c}{d_2} \tag{15}$$

which allows us to express the effective friction function, eq. (13) as:

$$\gamma\left(\boldsymbol{v}\right) = \gamma_0\, \frac{\left(\boldsymbol{v}^2 - \boldsymbol{v}_0^2\right)}{(q_0/\gamma_0) + \left(\boldsymbol{v}^2 - \boldsymbol{v}_0^2\right)} \tag{16}$$

We see that for $\boldsymbol{v} < \boldsymbol{v}_0$, i.e. in the range of small velocities, pumping due to *negative friction* occurs, as an additional source of energy for the Brownian particle (cf. Fig. 1) . Hence, slow particles are accelerated, while the motion of fast particles is damped. Due to the pumping mechanism introduced in our model, the conservation of energy clearly does not hold for the active particles, i.e. we now have a non-equilibrium, canonical-dissipative system (Feistel and Ebeling, 1989).

Negative friction plays an important role e.g. in technical constructions or in the theory of sound developed by Rayleigh (1945) already at the end of the last century. Here, the velocity-dependent friction function can be expressed as:

$$\gamma(\boldsymbol{v}) = -\gamma_1 + \gamma_2\,\boldsymbol{v}^2 = \gamma_1 \left(\frac{\boldsymbol{v}^2}{\boldsymbol{v}_0^2} - 1\right) \tag{17}$$

This Rayleigh-type model is a standard model for self-sustained oscillations studied also in the context of Brownian motion (Klimontovich, 1994). We note that $\boldsymbol{v}_0^2 = \gamma_1/\gamma_2$ defines here the special value where the friction function, eq. (17), is zero (cf. Fig. 1). Another example for a velocity dependent friction function with a zero $\boldsymbol{v}_0$ introduced in by Schienbein and Gruler (1993), reads:

$$\gamma(\boldsymbol{v}) = \gamma_0 \left(1 - \frac{\boldsymbol{v}_0}{\boldsymbol{v}}\right) \tag{18}$$

It has been shown that eq. (18) allows to describe the active motion of different cell types, such as granulocytes, monocytes or neural crest cells (Schienbein and Gruler, 1993; Gruler and Boisfleury-Chevance, 1994). Here, the speed $\boldsymbol{v}_0$ expresses the fact that the motion of cells is not only driven by stochastic forces, instead cells are also capable of self-driven motion.

Compared to eqs. (17), (18), the velocity-dependent friction function in eq. (13) is bound to a maximum value γ_0 reached for $\boldsymbol{v} \to \infty$ and avoids the singularity for $\boldsymbol{v} \to 0$ on the other hand (cf. Fig. 1).

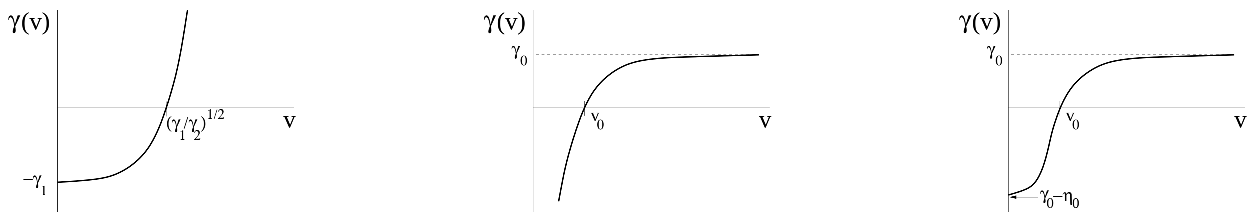


Figure 1: Different types of a velocity-dependent friction function, $\gamma(\boldsymbol{v})$ with a zero $\boldsymbol{v}_0$. (left) eq. (17), (middle) eq. (18), (right) eq. (13). For $\gamma(\boldsymbol{v}) < 0$ "pumping" dominates, while for $\gamma(\boldsymbol{v}) > 0$ "dissipation" dominates.

Provided a supercritical influx of energy, i.e. for $\boldsymbol{v}_0 > 0$, the passive motion of "usual" Brownian particles could be transformed into *active motion.* If we consider e.g. a two-dimensional space, the stationary velocity v_0 where the friction is just compensated by the energy supply, defines a cylinder in the four-dimensional space:

$$v_1^2 + v_2^2 = \boldsymbol{v}_0^2 \tag{19}$$

which attracts all deterministic trajectories of the dynamic system. This will be discussed in more detail in Sect. 4.2.

We note that instead of $\gamma_{\rm eff}$ also a more general friction function $\gamma(\boldsymbol{r},\boldsymbol{v})$ can be considered which may result in a similar dynamics. For example, Steuernagel *et al.* (1994) have discussed a space-dependent friction function, $\gamma(\boldsymbol{r})$, which can be possibly negative in a certain range of space: $\gamma(\boldsymbol{r}) < 0$ for $\boldsymbol{r} \in \mathcal{R}$. It has been shown that above a critical pumping, the particles may perform motion on a limit cycle, as well.

2.3 Stationary solutions for the distribution function

The existence of additional mechanisms of energy pumping will of course result in deviations from the known Boltzmann solution, eq. (5), for the distribution function $P(\boldsymbol{r},\boldsymbol{v},t)$ as will be discussed for the different types of non-linear friction functions introduced above. Let us consider a two-dimensional space without an external potential, i.e. $U(x_1,x_2) \equiv 0$.

The stationary solution of the Fokker-Planck eq. (4) reads for the friction function of the Rayleigh type, eq. (17)

$$P^0(\boldsymbol{v}) = C \exp\left[\frac{\gamma_1}{2D}\,\boldsymbol{v}^2 - \frac{\gamma_2}{4D}\,\boldsymbol{v}^4\right] \tag{20}$$

For the friction function, eq. (18) the stationary solution of the Fokker-Planck eq. (4) is of particular simplicity (Schienbein and Gruler, 1993)

$$P^0(\boldsymbol{v}) = C \exp\left[\frac{\gamma_0}{2D}\,(v - v_0)^2\right] \tag{21}$$

For the effective friction function, eq. (16), which is based on the effect of the internal energy depot, we find the stationary solution in the form (Erdmann *et al.*, 2000):

$$P^0(\boldsymbol{v}) = C' \left(1 + \frac{d_2\boldsymbol{v}^2}{c}\right)^{\frac{q_0}{2D}} \exp\left(-\frac{\gamma_0}{2D}\,v^2\right) \tag{22}$$

For a subcritical pumping of energy, $q_0 d_2 < c\gamma_0$, an *unimodal velocity distribution* results, centered around the velocity $\boldsymbol{v}_0 = 0$ like the *Maxwellian velocity distribution.* This is the case of the "low velocity" or *passive mode* for the stationary motion. However, for supercritical pumping, $q_0 d_2 > c\gamma_0$, a *crater-like velocity distribution* results. The corresponding "high velocity" or *active mode* (Schweitzer *et al.*, 2000) for the stationary motion is described by strong deviations from the Maxwell distribution.

In the limit of strong noise $D \sim T \to \infty$, i.e. at high temperatures, we get from eq. (22) the known Maxwellian distribution by means of eq. (3):

$$P^0(\boldsymbol{v}) = \left(\frac{1}{2\pi k_B T}\right) \exp\left(-\frac{\boldsymbol{v}^2}{2k_B T}\right) \tag{23}$$

which corresponds to standard Brownian motion in two dimensions. Hence, many characteristic quantities are explicitly known as e.g. the dispersion of the velocities

$$\langle \boldsymbol{v}^2 \rangle = 2k_B T \tag{24}$$

and the Einstein relation for mean squared displacement

$$\left\langle \Big(\boldsymbol{r}(t) - \boldsymbol{r}(0)\Big)^2 \right\rangle = 4\frac{k_B T}{\gamma_0}\,t \tag{25}$$

In the other limiting case of strong activation, i.e. relatively weak noise $D \sim T \to 0$ and/or strong pumping, we find a δ-distribution instead:

$$P^0(\boldsymbol{v}) = C\delta\left(\boldsymbol{v}^2 - \boldsymbol{v}_0^2\right)\ ;\quad \left\langle \boldsymbol{v}^2 \right\rangle = \boldsymbol{v}_0^2 \tag{26}$$

In order to treat this case in the full phase space, we follow Schienbein and Gruler (1993); Mikhailov and Meinköhn (1997) and introduce first an amplitude-phase representation in the velocity-space:

$$v_1 = v_0\,\cos(\phi)\ ;\quad v_2 = v_0\,\sin(\phi) \tag{27}$$

This allows us to separate the variables and we get a distribution function of the form:

$$P(x_1, x_2, v_1, v_2, t) = P(x_1, x_2, t)\cdot\delta(v_1^2 + v_2^2 - v_0^2)\cdot P(\phi, t) \tag{28}$$

The distribution of the phase ϕ satisfies the Fokker-Planck equation:

$$\frac{\partial}{\partial t}P(\phi, t) = D_\phi \frac{\partial^2}{\partial \phi^2}P(\phi, t) \tag{29}$$

By means of the known solution of eq. (29) we are able to calculate the mean square:

$$\left\langle \phi^2(t) \right\rangle = D_\phi t\ ;\quad D_\phi = \frac{D}{v_0^2} \tag{30}$$

where D_ϕ is the angular diffusion constant. By means of this, the mean squared spatial displacement $\left\langle r^2(t) \right\rangle$ of the particle can be calculated according to Mikhailov and Meinköhn (1997) as:

$$\left\langle r^2(t) \right\rangle = \frac{2v_0^4 t}{D} + \frac{v_0^6}{D^2}\left[\exp\left(-\frac{2Dt}{v_0^2}\right) - 1\right] \tag{31}$$

For times $t \gg v_0^2/D$, we find from eq. (31) the following expression for the effective spatial diffusion constant (Erdmann *et al.*, 2000):

$$D_r^{\mathrm{eff}} = \frac{2v_0^4}{D} = \frac{2}{D}\left(\frac{q_0}{\gamma_0} - \frac{c}{d_2}\right)^2 \tag{32}$$

where v_0, eq. (15), considers the additional pumping of energy resulting from the friction function, $\gamma(\boldsymbol{v})$. Due to this additional pumping, we obtain a high sensitivity with respect to noise expressed in the scaling with $(1/D)$. Moreover, just small changes in the direction of the active particles motion induce large excursion in space.

3 One-dimensional driven dynamics including forces

3.1 Motion in linear and ratchet-like potentials

In this section, the one-dimensional motion of active Brownian particles will be discussed. We denote the space coordinate by x. Further, the efficiency of energy conservation is assumed as $\eta = 1$. Moreover, we will neglect the possibility of environmental changes again. Then, the dynamics for the active Brownian motion is described by the following set of equations:

$$\begin{aligned} \dot{\boldsymbol{v}} &= -\big(\gamma_0 - d_2 e(t)\big)\boldsymbol{v} - \boldsymbol{\nabla} U + \sqrt{2D}\boldsymbol{\xi}(t) \\ \dot{e} &= q_0 - ce - d_2 \boldsymbol{v}^2 e \end{aligned} \tag{33}$$

Let us first discuss the most simple case of a linear external potential, $U(x) = ax$, hence the resulting force is a constant: $\boldsymbol{F} = -\boldsymbol{\nabla}\, U = -a$. The case of constant external forces was also treated by Schienbein and Gruler (1993) and Gruler and Boisfleury-Chevance (1994).

In the deterministic case, the stationary solutions of eq. (33) obtained from $\dot{\boldsymbol{v}} = 0$ and $\dot{e} = 0$ lead to a cubic polynom for the velocity $\boldsymbol{v}_0$:

$$d_2\gamma_0 \boldsymbol{v}_0^3 - d_2 \boldsymbol{F} v_0^2 - (q_0 d_2 - c\gamma_0)\boldsymbol{v}_0 - c\boldsymbol{F} = 0. \tag{34}$$

Depending on the value of F and in particular on the sign of the term $(q_0 d_2 - c\gamma_0)$, eq. (34) has either one or three real solutions for the stationary velocity, $\boldsymbol{v}_0$. This is also shown in the bifurcation diagram, Fig. 2.

The always existing solution expresses a direct response to the force in the form: $\boldsymbol{v}_0 \sim \boldsymbol{F}$, i.e. it results from the analytic continuation of Stokes' law, $\boldsymbol{v}_0 = \boldsymbol{F}/\gamma_0$, which is valid for $d_2 = 0$. We denote this solution as the "normal" mode of motion, since the velocity $\boldsymbol{v}$ has the same direction as the force $\boldsymbol{F}$ resulting from the external potential $U(x)$.

As long as the supply from the energy depot is small, we will also name the normal mode as the *passive mode*, because the particle is simply driven by the external force. More interesting is the case of three stationary velocities, $\boldsymbol{v}_0$, which significantly depends on the (supercritical) influence of the energy depot. In this case the particle will be able to move in a "high velocity" or *active mode* of motion. Fig. 2 shows that the former passive normal mode, which holds for subcritical energetic conditions, is transformed into an active normal mode, where the particle moves into the same direction, but with a much higher velocity. *Additionally*, in the active mode a new high-velocity motion *against* the direction of the force $\boldsymbol{F}$ becomes possible, which corresponds to an "uphill" motion.

It is obvious that the particle's motion "downhill" is stable, but the same does not necessarily apply for the possible solution of an "uphill" motion. Schweitzer *et al.* 2000 have investigated the

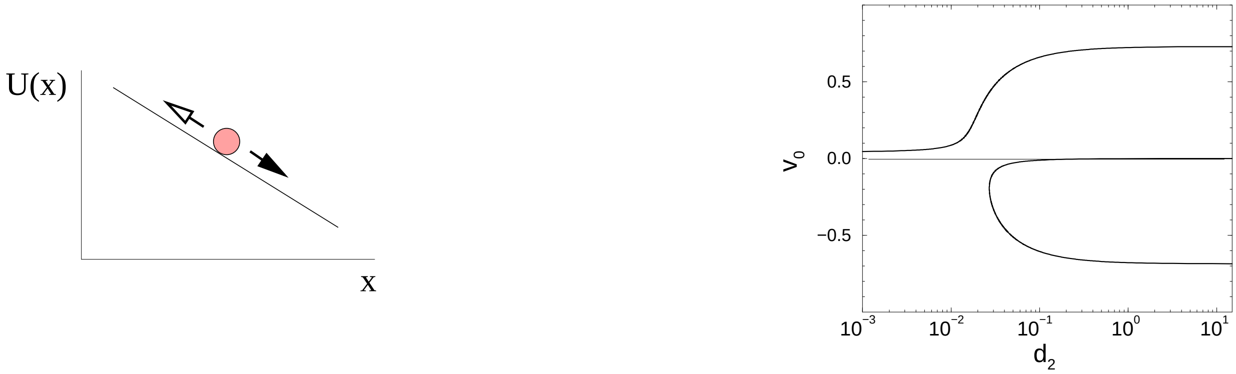


Figure 2: (left) Sketch of the one-dimensional motion of the particle in the presence of a constant force $\boldsymbol{F} = -\boldsymbol{\nabla} U(x)$ =const. Provided a supercritical amount of energy from the depot, the particle might be able to move "uphill", i.e. against the direction of the force. (right) Stationary velocities $\boldsymbol{v}_0$, eq. (34), vs. conversion rate d_2. Above a critical value of d_2, a negative stationary velocity indicates the posssibility to move against the direction of the force. Parameters: $\boldsymbol{F} = +7/8$, $q_0 = 10$, $\gamma_0 = 20$, $c = 0.01$. (Schweitzer *et al.*, 2000)

necessary conditions for such a motion in a linear potential. For the *deterministic* case, we found the following critical condition for a possible "uphill" motion of the pumped Brownian particle:

$$d_2^{crit} = \frac{F^4}{8q_0^3}\left(1 + \sqrt{1 + \frac{4\gamma_0 q_0}{F^2}}\right)^3 \tag{35}$$

In the limit of negligible internal dissipation $c \to 0$, eq. (35) describes how much power has to be supplied by the internal energy depot to allow a *stable uphill motion* of the particle.

This result will be used to explain the motion of active Brownian particles in a piecewise linear, asymmetric potential (cf. Fig. 3), which is known as a *ratchet potential* (Rousselet *et al.*, 1994; Luczka *et al.*, 1995).

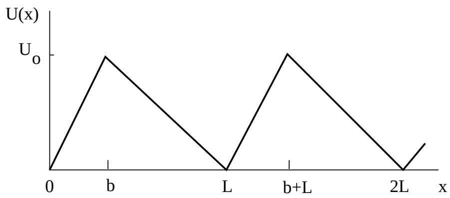


Figure 3: Sketch of the ratchet potential $U(x)$. For the computer simulations in Fig. 4 the following values are used: b=4, L=12, $U_0 = 7$ in arbitrary units.

In order to elucidate the class of possible solutions for the dynamics specified in eq. (33), let us discuss the phase-space trajectories for the *deterministic* motion, i.e $D = 0$.

Due to friction, a particle moving in the ratchet potential, will eventually come to rest in one of the potential wells, because the dissipation is not compensated by the energy provided from the internal energy depot. The series of Fig. 4 shows the corresponding attractor structures for the particle's motion dependent on the supply of energy expressed in terms of the conversion rate d_2. In Fig. 4a, we see that for a subcritical supply of energy expressed in terms of the conversion rate d_2 only *localized* states for the particles exist. The formation of limit cycles inside each minimum corresponds to stable oscillations in the potential well, i.e. the particles are not able to escape from the potential well.

With increasing d_2, the particles are able to climb up the potential flank with the lower slope, and this way escape from the potential well into negative direction ($d_2 > d_2^{crit1}$). As Fig. 4b shows, this holds also for particles which initially start into the positive direction. Thus, we find an unbound attractor corresponding to *delocalized motion* for negative values of v. Only if the conversion rate d_2 is large enough to allow the uphill motion along the flank with the steeper slope, the particles can escape from the potential well in *both* directions, and we find two unbound attractors corresponding to *delocalized* motion into both positive and negative direction ($d_2 > d_2^{crit2}$).

In the deterministic case, the particles will keep their direction determined by the initial conditions provided the energy supply allows them to move "uphill". In the stochastic case ($D > 0$), however, the initial conditions will be "forgotten" after a short time, hence due to stochastic influences, the particle's "upill" motion along the steeper flank will soon turn into a "downhill" motion. This motion into the negative direction will be most likely kept because less energy is needed. Thus, the stochastic fluctuations reveal the instability of an "uphill" motion along the steeper slope (Tilch *et al.*, 1999).

3.2 The Role of Interactions: Model of Dissipative Toda Chains

So far, we have not considered interactions between the active Brownian particles. Therefore, in this section, we consider an ensemble of N active particles located at the coordinates x_i on a ring with the total length L (cf. Fig. 5). These particles interact with their next neighbors via pair interactions of a rather special form, namely exponential repulsion. These so-called Toda interactions have the form:

$$U(r_i; \omega, b) = \frac{\omega^2}{b^2} \left(\exp\left[-b(r_i - \sigma)\right] - 1 + b(r_i - \sigma) \right) \tag{36}$$

where $r_i = x_{i+1} - x_i$ is the distance to the next particle, and σ is the equilibrium distance (cf. Fig. 5). The Toda interaction can be imagined as a spring; the parameter b is the stiffness of the spring and ω is the linear oscillation frequency around the equilibrium distance σ. We have assumed here that the average distance of the particles on the ring is equal to the equilibrium distance σ. This is not an essential restriction, since any change of the average distance may be compensated by a change of other parameters. For any choice of the parameters the global minimum of the potential corresponds to the equal distance of the particles.

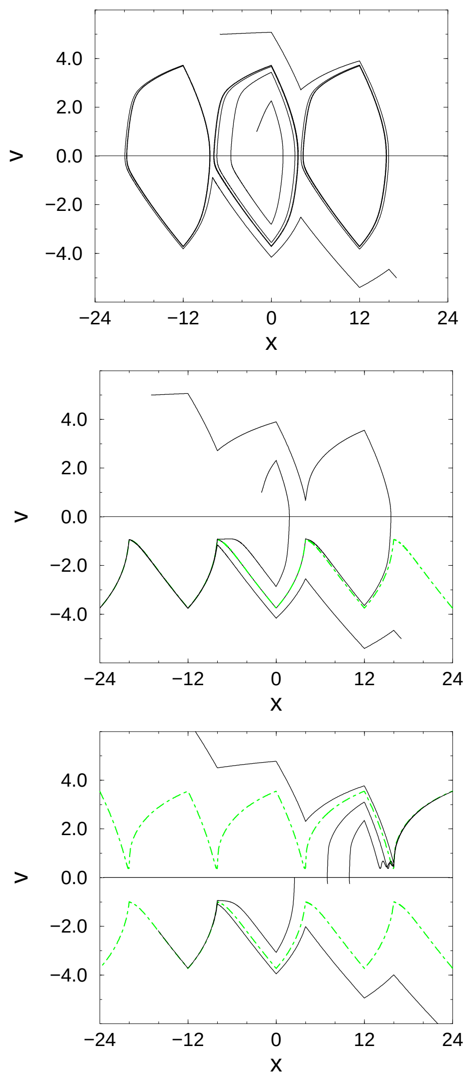
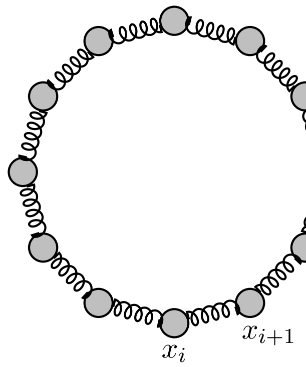


Figure 4: Phase-space trajectories of particles starting with different intitial conditions, for three different values of the conversion parameter d_2: (a: top) $d_2 = 1$, (b: middle) $d_2 = 4$, (c: bottom) $d_2 = 14$. Other parameters: $q_0 = 1$, $c = 0.1$, $\gamma_0 = 0.2$. The dashed-dotted lines in the middle and bottom part show the unbound attractor of the delocalized motion which is obtained in the long-time limit. (Tilch *et al.*, 1999)

One of the reasons for the special interest in Toda systems is the existence of exact solutions for the dynamics and the statistical thermodynamics. On this basis it was shown earlier that phonon excitations determine the spectrum at low temperatures and strongly localized soliton excitations are the most relevant at high temperatures. Ebeling and Jenssen (1991, 1992, 1999) have studied



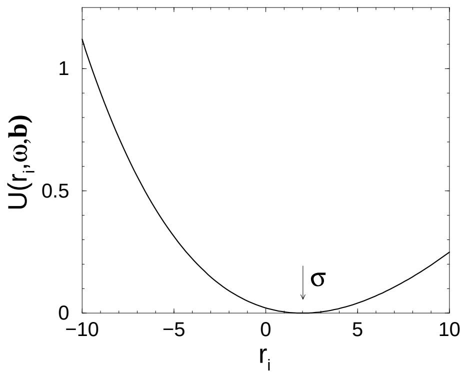


Figure 5: (left) Sketch of the one-dimensional Toda ring with N active Brownian particles. The interaction potential $U(r_i;\omega,b)$ between nearest neighbour particles is indicated by a spring. (right) Toda potential $U(r_i;\omega,b)$, eq. (36) vs. distance $r_i = x_{i+1} - x_i$. Parameters: $b = 0.1$, $\omega = 0.1$, $\sigma = 2.0$

several special effects in Toda rings with respect to noise and passive friction, the influence of non-uniformities and several temperature regimes. A first approach to investigate Toda rings of active particles with the aim to model dissipative solitons was given recently by Makarov *et al.* (2000). Here, we follow Ebeling *et al.* (2000) who investigated a model of dissipative Toda chains with noise and active (velocity-dependent) friction.

Using the interaction potential, eq. (36), the dynamics of an active Brownian particle $i = 1, ..., N$ is described by the following Langevin equation

$$\dot{\boldsymbol{v}}_i - A\left[e^{-b(x_i - x_{i-1})} - e^{-b(x_{i+1} - x_i)}\right] = -\gamma(v_i)v_i + \sqrt{2\,D}\boldsymbol{\xi}_i(t) \tag{37}$$

where $A = (\omega^2/b)\exp(b\,\sigma)$. The terms of the left-hand side of eq. (37) are of conservative nature, while the terms of the right-hand side are of dissipative origin. The last term describes again the fluctuations with strength D, whereas $\gamma(\boldsymbol{v}_i)$ denotes the non-linear friction function for which the velocity dependence of eq. (13) is assumed again. That means, we assume here a quasistationary energy depot, e_0, and the parameters η_0, η_2, eq. (14), and $\boldsymbol{v}_0$, eq. (15), characterize the pumping of energy.

It follows from eq. (15) that the condition $\eta_0 > \gamma_0$ has to be fulfilled for the active mode of motion. Consequently, the parameter

$$\alpha = \frac{\eta_0}{\gamma_0} - 1 \tag{38}$$

plays the role of the bifurcation parameter. Let us now study the solutions of eq. (37) with respect to α in the case of zero noise $D = 0$ (Ebeling *et al.* 2000). We have to distiguish between two different situations:

(i) For $\alpha < 0$, i.e. passive motion and only positive friction, $\eta_0 < \gamma_0$, the motion of all particles come to rest after a finite relaxation time which is of the order $1/\gamma(0)$. The dynamical system has only one attractor then, and any initial condition converges to the only stationary and stable solution

$$v_i = 0; \quad r_i = \sigma; \quad k = 1, 2, ..., N \tag{39}$$

This means that all particles are distributed on the ring at equal distances. The system has neutral stability with respect to a drift on the ring.

(ii) For $\alpha > 0$, we observe the appearance of $N+1$ coexisting attractors (Ebeling *et al.*, 2000), since the particles will move with a stationary non-zero velocity $\boldsymbol{v}_{0i}$. These attractors possess left-right symmetry with respect to rotations on the ring. The main difference between the attractor states is in qualitative respect given by different average velocities of the particles on the ring:

$$\langle v \rangle_t = \frac{1}{\tau} \int_0^\tau \frac{1}{N} \sum_{i=1}^{N} v_i(t)\, dt \tag{40}$$

where τ is the largest period of oscillations. The two left-right symmetrical attractors which correspond to the largest mean velocity are

$$v_i = \pm v_0\ ; \quad r_i = \sigma\ ; \quad k = 1, 2, ..., N \tag{41}$$

In this case, the particles are located at equal distances on the ring, their mean density being $\rho = N/L$, and rotate with the constant velocity v_0 either in clockwise or in counter-clockwise direction. This is a point attractor, the stability follows by an elementary analysis.

The remaining attractors correspond to excitations of two or more local compression pulses. Generally we can say that in areas where the nonlinear interaction forces are smaller than the pumping influence (e.g. around equilibrium distance) the particles aim to reach velocities $v_i = \pm v_0$. In areas dominated by the interaction, the particles are forced to slow down and finally change their directions of motion. In a first approximation, the combination of $(N-k)$ particles moving clockwise $(v_i > 0)$ and k counterclockwise prepares an attractor with a temporal mean of velocity per particle:

$$\langle v \rangle_t \approx \frac{N-2k}{N} \left(\frac{\alpha}{\eta_2}\right)^{\frac{1}{2}}. \tag{42}$$

It depends on the initial conditions, which attractor is eventually reached by the N particle system.

Numerical investigation of the deterministic system confirm the existence of these attractors (Ebeling *et al.*, 2000). In the case of strongly nonlinear interaction forces we observe for $k > 0$ different combinations of stationary soliton excitations. For a ring with $N =$ `odd` we find, in addition to the left-right constant rotations, $(N-1)$ attractors with non-zero average velocity, all having left-right symmetry. The two attractors with the second-largest average velocity are characterized by local compressions which are concentrated mostly on one of the springs and which are running left-right around the ring. This kind of excitation reminds on dissipative solitons. Such soliton-like excitations were investigated by Makarov *et al.* (2000) in a closely related model. The subsequent attractors show with decreasing average velocity, i.e. with incerasing k, more complicated compression patterns running left-right around the ring.

For $N = \texttt{even}$ a central attractor $k = N/2$ appears in addition to the attractors described above. It is characterized by a vanishing average velocity, i.e. all particles oscillate in a mode which reminds on the optical oscillation mode of lattices. Ebeling *et al.* (2000) have shown that simulations with stochastic initial conditions (random particle distribution and Gaussian velocity distribution) always lead to one of the attractors and give an idea of the attractor basins. Each attractor is characterized by a certain temporal mean of physical quantities, e.g. the mean energy per particle increases with the number k.

4 Active Brownian motion in two-dimensional potentials

4.1 Active Motion with localized energy sources

In the following, we discuss the particle's motion on a plane $\boldsymbol{r} = \{x_1, x_2\}$ under the influence of an external force $\boldsymbol{F} = -\boldsymbol{\nabla} U(\boldsymbol{r})$. While the case of constant forces was discussed in Sect. 3.1, we specify here the potential $U(\boldsymbol{r})$ as a parabolic potential:

$$U(x_1, x_2) = \frac{1}{2} a \, (x_1^2 + x_2^2) \tag{43}$$

Again, we neglect environmental changes of the particles, i.e. $h(\boldsymbol{r}, t) \equiv 0$. Further, also direct interactions between the active particles as discussed in the previous section, are neglected here. Then, the equations for the particle motion and the internal energy deopt are given again by eq. (33). However, in addition to the two-dimensional case, in this section we will also focus on a different aspect, namely the space-dependent take-up of energy, $q(\boldsymbol{r})$.

The parabolic potential, eq. (43), originates a force directed to the minimum of the potential. In a biological context, it simply models a “home”, and the moving object always feels a driving force pointing back to its “nest” (Ebeling *et al.*, 1999). Without an additional take-up of energy, the particle's position will fluctuate around the origin of the potential. For a supercritical supply, however, we find the motion of the particle on a stochastic limit cycle (Schweitzer *et al.*, 1998; Ebeling *et al.*, 1999; Erdmann *et al.*, 2000).

In the previous sections, we have always assumed a constant in space supply of energy, i.e. $q(\boldsymbol{r}) = q_0$. If, on the other hand, energy sources are localized in space, the internal depot of the active Brownian particles can be refilled only in a restricted area. This is reflected in a space dependence of the energy influx $q(\boldsymbol{r})$, for example:

$$q(x_1, x_2) = \begin{cases} q_0 & \text{if } \left[(x_1 - b_{1'})^2 + (x_2 - b_{2'})^2\right] \le R^2 \\ 0 & \text{else} \end{cases} \tag{44}$$

Here, the suppy area (*energy source*) is modeled as a circle, the center being different from the minimum of the potential. Noteworthy, the active particle is *not* attracted by the energy source due to long-range attraction forces. In the beginning, the internal energy depot is empty and active motion is not possible. So, the particle may hit the supply area because of the action of the stochastic force. But once the energy depot is filled up, it increases the particles motility, as presented in Fig. 6. Most likely, the motion into the energy area becomes accelerated, therefore an oscillating movement between the energy source and the potential minimum occurs after an initial period of stabilization.

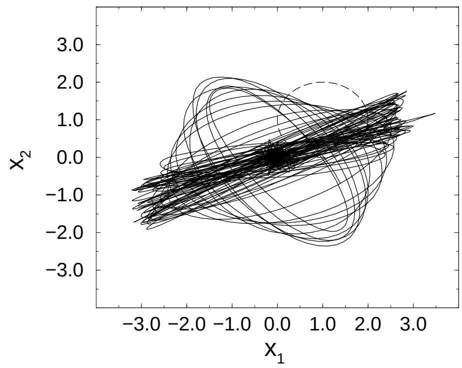


Figure 6: Trajectories in the x_1, x_2 space for the stochastic motion of the active Brownian particle in a parabolic potential, eq. (43) ($a = 2$). The circle (cooordinates (1,1), radius $R = 1$) indicates the area of energy supply, eq. (44). Parameters: $q_0 = 10$, $\gamma_0 = 0.2$, $c = 0.01$, $S = 0.01$, $d_2 = 0.1$, initial conditions: $(x_1, x_2) = (0, 0)$, $(v_1, v_2) = (0, 0)$, $e(0) = 0$. (Ebeling *et al.*, 1999)

Fig. 7 presents more details of the motion of the active Brownian particle, shown in Fig. 6. Considering the time-dependent change of the internal energy depot, the velocities and the space coordinates, we can distinguish between two different stages: In a first stage, the particle has not found the energy source, thus its energy depot is empty while the space coordinates fluctuate around the coordinates of the potential minimum. The second stage starts when the particle by chance, due to stochastic influences, reaches the localized energy source. Then the internal depot is soon filled up, which in turn allows the particle to reach out farther, shown in the larger fluctuations of the space coordinates. This accelerated movement, however, leads the active particle away from the energy source at the expense of the internal energy depot, which is decreased until the particle reaches the energy source again. Fig. 7 shows the corresponding oscillations in the energy depot.

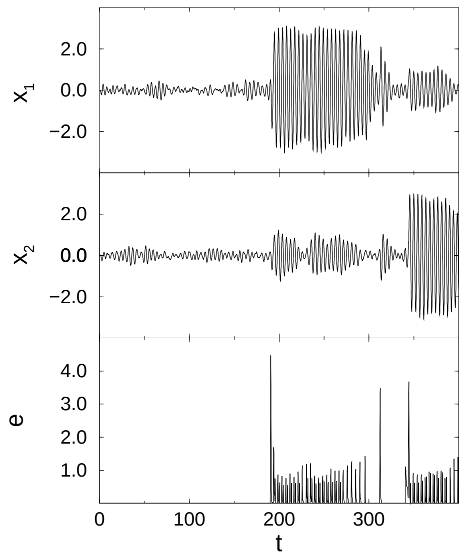


Figure 7: Space coordinates x_1, x_2 and internal energy depot e vs. time for the stochastic motion with $d_2 = 1$ shown in Fig. 6. (Ebeling *et al.*, 1999)

Interestingly, due to stochastic influences, the oscillating motion breaks down after a certain time, as shown in Fig. 7. Then the active particle, with an empty internal depot, moves again like a simple Brownian particle, until a new cycle starts. This way the particle motion is of intermittent type. Every new cycle starts with a *burst of energy* in the depot, indicated by the larger peak in $e(t)$, Fig. 7, which can be understood on the basis of eq. (7). We note that the transition time into the oscillation regime, as well as the duration of the oscillatory cycle depend remarkably on the conversion parameter, d_2 (Schweitzer *et al.* 1998). We also found that the trajectories eventually cover the whole area inside certain boundaries, however during an oscillation period the direction is most likely kept.

The motion of active Brownian particles in two-dimensional landscapes with localized energy sources could be extended to describe more complex situations. We just want to mention the case of many separated potential minima of $U(\boldsymbol{r})$. On the other hand, we can also assume many separated energy sources, randomly distributed on the surface, which means a specification for $q(\boldsymbol{r})$. We can also generalize the situation by considering that the availablility of energy is both space and time dependent, which means additional conditions for the take-up of energy, $q(\boldsymbol{r}, t)$. Let us consider a situation, where the supplied external energy ("food") grows with a given flow density $\Phi(\boldsymbol{r}) = \eta q_f(\boldsymbol{r})$, with η being a dimensional constant. Then, we assume that the change of the energy take-up function may depend both on the increase and the decrease in supplied energy, i.e:

$$\dot{q}(\boldsymbol{r}, t) = \Phi(\boldsymbol{r}) - \eta q(\boldsymbol{r}, t) = \eta \Big[q_f(\boldsymbol{r}) - q(\boldsymbol{r}, t) \Big] \tag{45}$$

The formal solution for the energy take-up yields:

$$q(\boldsymbol{r},t) = e^{-\eta t} q(\boldsymbol{r},0) - \eta \int_{-\infty}^{t} d\tau \, e^{-\eta(t-\tau)} q_f\Big(\boldsymbol{r}(\tau),\tau\Big) \tag{46}$$

As we see, the actual value of the energy influx now depends on the whole prehistory of the motion of the particle and reflects a certain kind of memory. However, those memory effects will be neglected here, which means $\eta \to \infty$.

We may also consider that active motion occurs in more complex landscapes which typically not only contain localized areas of energy supply, but also *obstacles*. We have discussed such a case while assuming a hard-core like obstacle where the particle is simply reflected at the boundary if it hits the obstacle. Considering a continuous supply of energy but only a deterministic motion, Schweitzer *et al.* (1998) found a chaotic motion of the active particle in the phase space $\Gamma = \{x_1, x_2, v_1, v_2, e\}$. Hence, we concluded that for the motion of Brownian particles with energy depots reflecting obstacles have an effect similar to stochastic influences (external noise).

4.2 Motion of "swarms"

Eventually, we may consider again the motion of an ensemble of N active Brownian particles. The case of direct interactions between the particles (cf. Sect. 3.1) will be neglected here, further, we may assume a constant take-up of energy, $q(\boldsymbol{r}) = q_0$. Thus, the question arises under which conditions a kind of coordinated motion of the particles can be observed in a two-dimensional parabolic potential.

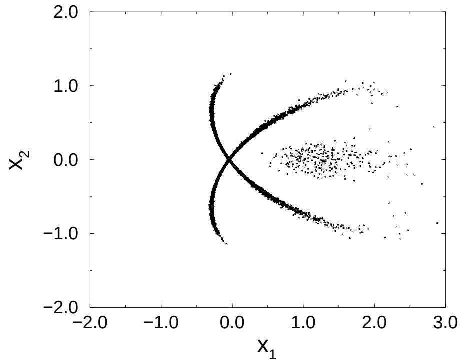
Fig. 8 shows two snapshots of an ensemble of active particles. Initially, all particles start with an empty internal energy depot. Further, in this particular simulation, the initial position was the same for all particles and different from the potential minimum. Thus, for intermediate times, the motion of the particle ensemble reminds on *swarming*, i.e. a coherent motion with slow spatial dispersion (Schweitzer *et al.*, 2001). After an inital stage, we find the occurence of two branches of the swarm which results from a symmetry break (cf. Fig. 8 a). These two branches will, after a sufficient long time, move on two limit cycles (as already indicated in Fig. 8 b). One of these limit cycles refers to the left-handed, the other one to the right-handed direction of motion in the 2d-space.

The radius of the limit cycles obeys the relation (Erdmann *et al.*, 2000):

$$x_1^2 + x_2^2 = r_0^2 = \frac{v_0^2}{a} = \text{const.} \tag{47}$$

where v_0 is the stationary velocity of the particles on the limit cycle. It has been shown by Ebeling *et al.* (1999) that for the case of a parabolic potential, v_0 has the same value as for the force-free case ($U = \text{const.}$) eq. (15), namely:

$$v_1^2 + v_2^2 = \boldsymbol{v}_0^2 = \frac{q_0}{\gamma_0} - \frac{c}{d_2} = \text{const.} \tag{48}$$



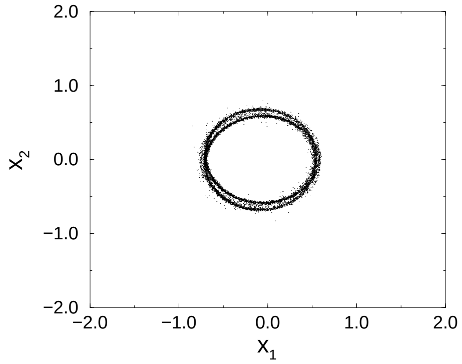


Figure 8: Snapshots of a swarm of $N = 2000$ active Brownian particles moving in a two-dimensional potential, eq. (43). (top) $t = 15$, (bottom) $t = 99$. Initial conditions for all particles: $x_1 = 0.0$, $x_2 = 0.5$, $v_1 = 0.0$, $v_2 = 0.0$, $e(0) = 0$. Parameters: $q_0 = 10$; $c = 1.0$; $\gamma_0 = 20$, $d_2 = 10$. (Schweitzer *et al.*, 2001)

We note that the swarm-like motion of the particles, which eventually reaches a limit cycle, will only occur for a supercritical supply of energy from the depot, i.e. for $\boldsymbol{v_0^2 > 0}$. The total mechanical energy of an active particle moving on the limit cycle can then be expressed as:

$$E_0 = \frac{1}{2}(v_1^2 + v_2^2) + \frac{a}{2}(x_1^2 + x_2^2) = \frac{1}{2}v_0^2 + \frac{a}{2}r_0^2 \tag{49}$$

Ebeling *et al.* (1999) have shown that any initial value of the energy converges (at least in the limit of strong pumping) to

$$H \longrightarrow E_0 = v_0^2 \tag{50}$$

This corresponds to an equal distribution between kinetic and potential energy, i.e. similar to the harmonic oscillator in one dimension, both parts contribute the same amount to the total energy.

A closer inspection of the limit cycles (Erdmann *et al.*, 2000) shows that the trajectories of the particles are like a hoop in the four-dimensional space $\{x_1, x_2, v1, v_2\}$. Most projections to the two-dimensional subspaces are circles or ellipses, as shown above, however there are two subspaces, namely $\{x_1, v_2\}$ and $\{x_2, v_1\}$ where the projection is like a rod. A second limit cycle is obtained by time reversal, $t \to -t,\ v_1 \to -v_1,\ v_2 \to -v_2$. This limit cycle also forms a hula hoop which is different from the first one in that the projection to the $\{x_1, x_2\}$ plane has the opposite rotation direction compared to the first one. However both limit cycles have the same projections to the $\{x_1, x_2\}$ and to the $\{v_1, v_2\}$ plane. The separatrix between the two attractor regions is given by the plane $(v_1 + v_2 = 0)$ in the four-dimensional space.

In the *stochastic* case (Erdmann *et al.*, 2000) we find that the two hoop rings are converted into a distribution looking like two embracing hoops with finite size, which for strong noise convert into two embracing tires in the four-dimensional space. While in the deterministic case either left- or righthanded rotations are found, in the stochastic case the system may switch randomly between the left- and righthand rotations, since the separatrix becomes transparent.

5 Discussion and applications

Active motion is a phenomenon found in a wide range of systems. For instance, biological motion, i.e. the motion of organisms or cells, can be active and passive (Alt and Hoffmann, 1990). Passive motion can be driven by convection, currents, external fields, or even by thermal fluctuations if the organism is microscopically small. Active motion, on the other hand, occurs under energy consumption and requires metabolic activity and a supply of fuel.

A possible physical approach to active motion is based on active particles, which are resently used for modelling several types of complex motion. On the physico-chemical level, for example, Brownian particles with chemical interactions were studied by Nossal (1983), Mikhailov and Meinköhn (1997) and by Schimansky-Geier *et al.* (1995, 1997).

Among the phenomena presently investigated in cell biology is the directed movement of "particles" (e.g. kinesin or myosin molecules) along periodic structures (e.g. microtubules or actin filaments) (Finer, 1994) in the *absense of a macroscopic force*, which may have resulted from temperature or concentration gradients. In order to reveal the microscopic mechanisms resulting in directed movement, different physical ratchet models based on Brownian particles have been proposed (Magnasco, 1993; Maddox, 1994; Millonas and Dykman, 1994; Rousselet *et al.*, 1994; Luczka *et al.*, 1995; Derenyi and Vicsek, 1995). These models have in common to transfer the undirected motion of Brownian particles into a directed motion, therefore the term "Brownian rectifiers" has been established. Applications to molecular motors were studied by Magnasco (1994), Astumian and Bier (1994), Jülicher and Prost (1995), Hänggi and Bartussek (1996).

Further, active particle models were used to describe the motion of cells (Schienbein and Gruler, 1993; Gruler and Boisfleury-Chevance, 1994), or the motion of macroscopic biological objects like insects (Deneubourg *et al.*, 1990, Calenbuhr and Deneubourg, 1991; Schweitzer *et al.* 1997), or the coherent motion of animals like birds (Viscek *et al.*, 1995). Eventually, we also mention related applications to the motion of pedestrians (Helbing *et al.*, 1997) and to traffic problems (Wolf *et al.*, 1996; Helbing, 1997; Mahnke and Pieret, 1997; Helbing and Huberman, 1998).

The main objective of the work summarized in this paper, is to study active motion which relies on energy supply from the environment. Further, mechanisms of energy take-up, storage and conversion are taken into account. In order to derive a simple approach for these features, we have suggested a model of active Brownian particles, which are Brownian particles with an internal energy depot as a new degree of freedom. The basic equations of this approach are an extended Langevin eq. (8), which is coupled to the balance eq. (7) for the internal energy depot. The depot energy can be used to perform different activities, such as metabolism, acceleration of motion, changes of the environment, or signal-response behavior. This way, the particles become more complex, which result in new dynamical features, such as:

1. non-linear, velocity dependent friction functions (cf. Sect. 2.2),
2. non-equilibrium velocity distributions with crater-like shape (cf. Sect. 2.3),
3. new diffusive properties with large mean square displacement (cf. Sect. 2.3),
4. uphill motion and directed transport in periodic potentials (cf. Sect. 3.1),
5. excited collective motion of interacting active particles (cf. Sect. 3.2),
6. intermittent (active/passive) "behavior" and directed motion in the presence of localized energy sources (cf. Sect. 4.1),
7. the formation of limit cycles corresponding to left/righthand rotations (cf. Sect. 4.2).

We note that some of these features may resemble active biological motion, although we point out again that we did *not* intend to model a particular system. Instead, the basic idea of our approach can be formulated as follows: How much of physics is needed to achieve a degree of complexity which gives us the impression of phenomena found in biological or technical systems?

In this paper, we have studied only some of the physical aspects with respect to active motion, such as mechanisms of energy pumping and energy dissipation. These investigations should contribute to the development of a microscopic theory of active biological motion. Moreover, modifications of our model may also be used to simulate coherent motion of higher organisms, or traffic problems.

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