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Social impact models of opinion dynamics

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Abstract

We investigate models of opinion formation which are based on the social impact theory. The following approaches are discussed: (i) general mean field theory of social impact, (ii) a social impact model with learning, (iii) a model of a finite group with a strong leader, (iv) a social impact model with dynamically changing social temperature, (v) a model with individuals treated as active Brownian particles interacting via a communication field.

1 Introduction

In recent years, there has been a lot of interest in applications of physical paradigms for a *quantitative* description of social and economic processes [1–12]. These attempts usually raise controversial discussions. From the perspective of the life and social sciences, one is afraid of an unjustified reduction of the complex relations in socio–economical systems, in order to fit them into a rather "mechanical" description [9]. From the perspective of physics, on the other hand, one claims that the description of such processes "evidently lies outside of the realm of physics" (to quote an unknown referee, not for this manuscript).

Despite these objections, the development of the interdisciplinary field "science of complexity" has lead to the insight that complex dynamic processes may also result from simple interactions, and even *social* structure formation could be well described within a mathematical approach. This is not an artifact. Statistical mechanics is meant to comprise any phenomena where the relationship between microscopic properties and macroscopic behavior plays a role. The problem, however, is to understand carefully the reductions regarding the system elements and their interactions, when turning to socio–economical systems. For example, one is usually confronted with individuals capable of mental reflections and purposeful actions, creating their own reality, and the question, how this interferes with a rather autonomous or "self-organized" social dynamics is far from being solved.

Nevertheless, a broad range of *dynamical* methods originally developed in a *physical* context have been successfully applied to socio-economic phenomena. For instance, economical models have been extensively studied using the techniques of stochastic dynamics [10], percolation theory [11] or the chaos paradigm [12]. Another important subject of this kind is the process of *opinion formation* treated as a collective phenomenon. On the "macroscopic" level it can be described using the master equation or Boltzmann-like equations for global variables [2,4,8,13,14], but microscopic models are constructed and investigated as well [15,16] using standard methods of statistical physics.

A quantitative approach to the dynamics of opinion formation is related to the concept of *social impact* [17–23], which enables to apply the methods similar to the cellular automata [24]. The aim of this review is to revise various models of opinion formation that are based on the social impact theory. Sec. 2 is devoted to general properties of the social impact model studied within a mean field approach [20]. Sec. 3 investigates the influence of learning processes on the final phases of social impact models. In Sec. 4 we consider phase transitions in a social impact model that can occur in a finite group in the presence of a *strong individual* (a leader) [25–27]. As two special cases, we discuss a purely deterministic limit and a noisy model. In Sec. 5, we discuss a social impact model where the social temperature is a dynamical variable coupled to changes of the global group opinion. Sec. 6 is devoted to an extension of social impact models to include phenomena of migration, memory effects and a finite velocity of information exchange. Here the concepts of *active Brownian particles* [9–12, 28–30] and the communication field [31] will be applied.

2 Nowak-Szamrej-Latané models

A class of models of opinion formation based on the concept of cellular automata has been proposed by Nowak, Szamrej and Latané [21]. The sociological basis of these models is the theory of *social impact* formulated by Latané [17] who claimed that the impact exerted on an individual by a group of people is a multiplicative function of their social immediacy, strength and number. Meanwhile, a large empirical support to this statement has been gathered [18, 19]. Here the formulation and properties of the models are briefly recalled [20].

The model group consists of N individuals, each of them can hold one of two opposite opinions $\sigma_i = \pm 1$. This is relevant not only to typical "yes"-"no" questions, but also to important issues where the distribution of opinions seems to be bimodal, peaked on extreme values. Every individual is characterized by two strength parameters: persuasiveness p_i and supportiveness s_i, describing the strength of interactions with individuals holding opposite or the same opinions. These parameters are assumed to be random numbers with the mean values $\overline{s}$ and $\overline{p}$ respectively, this way introducing disorder in the system and allowing a complex dynamics in contrast to plain Ising models. The individuals are located in *social space*, each pair is ascribed a distance d_{ij}; the magnitude of mutual interactions decreases with the distance. The dynamics of the opinion changes is given by the rule

$$\sigma_i(t+1) = -\mathrm{sign}(\sigma_i(t) I_i(t) + h_i) \tag{1}$$

applied synchronously to every individual. The quantity I_i is the social impact given as the sum of influences on individual i from all other individuals. Positive influences arise from those sharing the opposite opinion and negative from those sharing the same opinion:

$$I_i = I_p \left(\sum_{j=1}^{N} \frac{t(p_j)}{g(d_{ij})} (1 - \sigma_i \sigma_j) \right) - I_s \left(\sum_{j=1}^{N} \frac{t(s_j)}{g(d_{ij})} (1 + \sigma_i \sigma_j) \right) \tag{2}$$

Here g is an increasing function of the distance d_{ij}, t is the strength scaling function, and I_p, I_s are the impact form functions. The parameter $\beta = 1/g(d_{ii})$ is a so called self-supportivness. The additional term h_i can be a random variable introducing noise to the system; it can also describe a general preference towards one of the opinions.

Extensive computer simulations have been performed [21] for the model using Euclidean geometry with $g(d_{ij}) = d_{ij}^{\alpha}$, $2 < \alpha \leq 8$, uniformly distributed strength parameters, the scaling function $t(x) = x$ and various forms of impact functions e.g. $I(x) = \sqrt{x}$ or $I(x) = x$. The most dominant phenomena that have been observed are the *clustering* and *polarization* of opinions. Starting from some random distribution of opinions the system converges quickly to a stationary state in which the minority has shrunk with respect to the initial number and their members are grouped in clusters (the clustering proceeds due to changes of opinion only, movements of individuals in social space are not included). In the presence of noise the clusters appear to be metastable states; they remain stationary for some time, then suddenly shrink to some other clustered minority state which in turn persists for another relatively long time period. This kind of behaviour has been called a *staircase dynamics*. In the case of unbounded noise the only globally stable state is the unification of opinions, however for relatively low noise intensities the states of the clustered minority can remain stationary for an exponentially long time.

The results of computer simulations have been supported by a theoretical analysis of the model based on a mean field-like approach [17]. Different types of geometry have been considered: (i) a fully connected model, where $g(d_{ij}) = N$ for all $i \neq j$, (ii) strongly diluted connections, on the other hand, (iii) a hierarchical geometry, in which the distance between individuals belonging to the same subgroup are identical, and (iv) an Euclidean geometry. The essence of the theory is to choose an appropriate order parameter reflecting the complexity of the system, which is averaged over the quenched disorder (it no longer depends on random parameters s_i and p_i), and to derive from (1) an equation describing equivalently the time evolution of the order parameter. Fixed points of this equation correspond to stationary states. In the presence of noise the stability of the solutions against perturbations should be additionally considered.

For example in the case of the fully connected model, i.e. when $g(d_{ij}) = N$ for all $i \neq j$ the order parameter $n(\xi)$ is a function of a variable $\xi \in [0, \infty]$:

$$n(\xi) = \left\langle \frac{1}{N} \sum_{j=1}^{N} \frac{s_j + p_j}{\overline{s} + \overline{p}} \Theta(a_j - \xi) \right\rangle \; ; \;\; a_i = \frac{\overline{s} - \overline{p}}{\overline{s} + \overline{p}} + \frac{\beta s_i}{\overline{s} + \overline{p}} \tag{3}$$

where $\Theta(x)$ is the Heaviside function. The parameters a_i describe the effective self-supportiveness and the brackets $< >$ denote the average over the quenched disorder (random variables s_i and p_i). ¿From the dynamical rule (1), in the absence of noise ($h_i = 0$), one can derive an equation for evolution of $n(\xi)$:

$$n'(\xi) = [g(m, \xi) + n(|m|)]\Theta(|m| - \xi) + n(\xi)\Theta(\xi - |m|), \tag{4}$$

where $m = n(0)$ is a weighted majority-minority difference ($m \in [-1, 1]$), and

$$g(m, \xi) = \left\langle \frac{1}{N} \text{sign}(m) \sum_{j=1}^{N} \frac{s_j + p_j}{\overline{s} + \overline{p}} \Theta(|m| - a_j)\Theta(a_j - \xi) \right\rangle . \tag{5}$$

Further, the map describing the dynamics of m follows from (4):

$$m' = g(m, 0) + n(|m|). \tag{6}$$

It can be shown that $|m|$ is an increasing and bounded function of the time step so it has at least one stable fixed point. Actually in a generic case it has many fixed points, separated by unstable ones, corresponding to many possible stationary states. In the presence of noise the fixed points form a kind of sequence of metastable states with increasing $|m|$. The system remains in subsequent states for a long time interrupted by noise induced jumps to the next state (staircase dynamics). If the noise level is small, the residence times in the metastable states may be extremely long.

A similar approach can be extended for the case of a hierarchical geometry. The strongly diluted model with randomly changing connections is shown to be equivalent to the fully connected one in the limit of small noise. For all these geometries the staircase dynamics remains the persistent feature of the system.

3 Learning effects in social impact theory

Kohring [23] has considered a model of opinion formation where parameters describing interactions between members of the social group can change in time, which would correspond to a learning procedure. He has assumed that the opinion dynamics is given by

$$\sigma_i(t+1) = \sigma_i(t)\ \mathrm{sign}\left(\sum_{j=1}^{N} S_{ij}(1+\sigma_i\sigma_j) - \sum_{j=1}^{N} P_{ij}(1-\sigma_i\sigma_j)\right) \tag{7}$$

where the positive parameters S_{ij} and P_{ij} describe the strength of social influence of the individual j on the individual i. Now, one can apply the principle of *cumulative advantage* by Price [34] saying that the increase of one's performance is proportional to the current level of performance. It follows that the simple learning rule can be introduced as

$$S'_{ij} = \begin{cases} S_{ij} + \alpha_s S_{ij} & \text{if} \quad \sigma_j(0) = \sigma_j(F) = \sigma_i(F) = \sigma_i(0) \\ S_{ij} & \text{otherwise} \end{cases} \tag{8}$$

and

$$P'_{ij} = \begin{cases} P_{ij} + \alpha_p P_{ij} & \text{if} \quad \sigma_j(0) = \sigma_j(F) = \sigma_i(F) \neq \sigma_i(0) \\ P_{ij} & \text{otherwise} \end{cases} . \tag{9}$$

Here $\sigma_i(0)$ and $\sigma_i(F)$ are the mean initial and the mean final state, while α_s and α_p are constant parameters that determine the learning speed. Numerical simulations have shown that there is a large difference between the case of low speed learning, $\alpha_s = \alpha_p \to 1$, and the case of high learning rates, $\alpha_s = \alpha_p \approx 1.3$. The first case corresponds to the model without learning and is characterized by large values of the final mean opinion

$$m = \frac{1}{N}\sum_{i=1}^{N} \sigma_i(F) \tag{10}$$

and by normally distributed correlations of the initial and final values of individual opinions $d(i) = \langle \sigma_i(0)\sigma_i(F)\rangle$, which can be attributed to the presence of a *ferromagnetic state*. The second case is characterized by low values of the final mean opinion m and by large values of correlations $d(i)$, which can be attributed to a *frozen spin–glass state*

Apart from the learning rules given by (8,9) Kohring [23] has also studied the example $P_{ij} = S_{ij}$ where

$$S'_{ij} = \begin{cases} S_{ij} + \alpha S_{ij} & \text{if} \quad \sigma_j(0) = \sigma_j(F) = \sigma_i(F) \\ S_{ij} & \text{otherwise} \end{cases} \tag{11}$$

The above learning rule leads to large values of the mean opinion m even for large learning rates $\alpha \leq 10$, i.e. the ferromagnetic order in the final state has been found. If one studies the correlation function

$$c(i) = \frac{1}{N}\sum_{j=1}^{N} \langle \sigma_j(F)\sigma_i(0)\rangle \tag{12}$$

between the initial value of the individual opinion σ_i and the final values of all other opinions, the difference between lower and higher learning rates can be observed. Computer simulations have shown that for larger values of α most of the final opinions are highly correlated with the initial opinion of a *single* individual or with those of a small number of individuals. Thus a single individual or a few individuals determine the final opinion of the *whole group*, a phenomenon which can be interpreted as the emergence of *leaders* in such a model.

4 Phase transitions in the presence of a strong leader

4.1 The model

In [25–27] Kacperski and Hołyst have studied a special case of social impact model namely when a strong individual (leader) is present in a social group. Similarly as in Sec.2 the system consists of N individuals sharing the opinions $\sigma_i = \pm 1$, $i = 1, 2, ...N$. Each individual is characterised by the parameter $s_i > 0$ which describes the strength of his/her influence. The strength parameters s_i of the individuals are positive random numbers with the probability distribution $q(s)$ and the mean value $\overline{s}$. We assume that our social space is a two-dimensional disc of radius $R \gg 1$, with the individuals located on the nodes of a quadratic grid. The distance between nearest neighbours equals 1, while the geometric distance models the social immediacy. In the centre of the disc there is a *strong individual* (who we will call the "leader"); his/her strength s_L is much larger than that of all the others ($s_L \gg s_i$).

Changes of opinions are determined by the following social impact exerted on every individual:

$$I_i = -s_i\beta - \sigma_i h - \sum_{j=1, j\neq i}^{N} \frac{s_j \sigma_i \sigma_j}{g(d_{ij})}, \tag{13}$$

where $g(x)$ is an increasing function of the social distance d_{ij}. β is a self–support parameter reflecting the inclination of an individual to maintain his/her current opinion. h is an additional (external) influence which may be regarded as a global preference towards one of the opinions stimulated by mass–media, government policy, etc.

Opinions of individuals may change simultaneously (synchronous dynamics) in discrete time steps according to the rule:

$$\sigma_i(t+1) = \begin{cases} \sigma_i(t) & \text{with probability} \quad \dfrac{\exp(-I_i/T)}{\exp(-I_i/T) + \exp(I_i/T)} \\ -\sigma_i(t) & \text{with probability} \quad \dfrac{\exp(I_i/T)}{\exp(-I_i/T) + \exp(I_i/T)} \end{cases} . \tag{14}$$

Eq. (14) is analogous to the Glauber dynamics with $-I_i\,\sigma_i$ corresponding to the local field. The parameter T may be interpreted as a "social temperature" describing a degree of randomness in the behaviour of individuals, but also their average volatility (cf [16]). The impact I_i is a "deterministic" force inclining the individual i to change his/her opinion if $I_i > 0$, or to keep it otherwise. The model is a particular case of the system considered in [20].

4.2 Deterministic limit

Let us first recall the properties of the system without noise, i.e. at $T = 0$ [25, 26]. Then, the dynamical rule (14) becomes strictly deterministic:

$$\sigma_i(t+1) = -\mathrm{sign}(I_i \sigma_i). \tag{15}$$

Considering the possible stationary states we find either the trivial unification (with equal opinion ± 1 for each individual) or, due to the symmetry, a circular cluster of individuals who share the opinion of the leader. This cluster is surrounded by a ring of their opponents (the majority). These states remain stationary also for a small self-support parameter β; for sufficiently large β any configuration may remain "frozen".

Using the approximation of a continuous distribution of individuals (i.e. replacing the sum in (13) by an integral) one can calculate the size of the cluster, i.e. its radius a as a function of the other parameters. In the case of $g(r) = r$ and $\overline{s} = 1$ we get from the limiting condition for the stationarity $I = 0$ at the border of the cluster:

$$a \approx \frac{1}{16}\left[2\pi R - \sqrt{\pi} \pm \beta - h \pm \sqrt{(2\pi R - \sqrt{\pi} \pm \beta - h)^2 - 32 s_L}\,\right]. \tag{16}$$

This is an approximate solution valid for $a \ll R$, but it captures all the qualitative features of the exact one which can be obtained by solving a transcendent equation (cf. Fig. 1). Here and in the next section we assume that the leader's opinion is $\sigma_L = +1$, but the analysis is also valid for the opposite case if $h \to -h$.

The branch with the "−" sign in front of the square root in Eq. (16) corresponds to the stable cluster. The one with "+" corresponds to the unstable solution which separates the basins of attraction of the stable cluster and unification (cf. Fig. 1). Owing to the two possible signs at the self–support parameter β in (16), the stable and unstable solutions are split and form in fact two bands. The states within the bands are "frozen" due to the self–support which may be regarded as an analogy of the dry friction in mechanical systems. This way also the unstable clusters can be observed for $\beta > 0$ and appropriately chosen initial conditions.

According to Eq. (16) real solutions corresponding to clusters exist provided

$$(2\pi R - \sqrt{\pi} \pm \beta - h)^2 - 32 s_L \geq 0. \tag{17}$$

Otherwise the general acceptance of the leader's opinion (unification) is the only stable state. When, having a stable cluster, the condition (17) is violated by changing a parameter e.g. s_L or h, one can observe a discontinuous phase transition: *cluster* $\to$ *unification*.

If, on the other hand, the leader's strength is too weak, it may be impossible for him/her not only to form a cluster but also to maintain his/her own opinion. The limiting condition for the minimal leader's strength s_{Lmin} to resist against the persuasive impact of the majority can be calculated from the limiting condition $I_L = 0$ (I_L - the impact exerted on the leader):

$$s_{Lmin} = \frac{1}{\beta}(2\pi R - \sqrt{\pi} - h). \tag{18}$$

Fig. 1 shows a phase–diagram s_L-a for $h = 0$. All the plots are made for a space of radius $R = 20$ (1257 individuals) and $\beta = 1$ unless stated otherwise. Points in Fig. 1 are obtained by numerical simulations of (15) while the curves are solutions of a transcendent equation following from the stationary condition $I(a) = 0$. Solid lines represent stable fixed points – attractors (they correspond to the solution (16) with "−" sign before the square root); dashed lines represent unstable repellers (corresponding to "+" in (16)).

We find two kinds of attractors: (i) unification ($a = R$ when the leader's opinion wins, $a = 0$ when it ceases to exist) and (ii) a stable cluster resulting from a solution of (16). In the s_L-a space one can distinguish between three basins of attraction. Starting from a state in the area denoted as I, the time evolution leads to unification with $a = 0$. The stable cluster attractor divides its basin of attraction into the areas IIa and IIb. All states from III will evolve to unification with $a = 20$. Owing to the two possible signs of the self–support parameter β in (16), the attractor and repeller are split. The space between their two parts enclose the "frozen" states that do not change in the course of time. These states correspond to local equilibria of the system dynamics similar to spin glass states. Thus, as a result of self–support, even repeller states can be stabilized. As one can see, the results of computer simulations fit the calculated curves very well.

Taking into account the conditions (17), (18) and the two possible opinions of the leader one can draw a phase-diagram $h - s_L$ distinguishing the regions where different system states are possible [25, 27]. Apparently, the system shows *multistability* in a certain range of s_L and h. It depends on the history which of the states is realized, so we can observe a *hysteresis* phenomenon [25, 27]. Moving in the parameter space $s_L - h$, while starting from different configurations one can have many possible scenarios of phase transitions [27].

4.3 Effects of social temperature

It is obvious that the behaviour of an individual in a group depends not only on the influence of others. There are many more factors, both internal (individualal) and external, that induce opinion formation and should be modeled somehow. In our model, we do this by means of a noisy dynamics, i.e. we use equation (14) with the parameter $T > 0$. In the presence of noise, the marginal stability of unstable clusters due to the self-support is suppressed and they are no longer the stationary states of the system. The borders of the stable clusters become diluted, i.e. individuals of both opinions appear all over the group. Our simulations [25, 27] prove that the presence of noise can induce the transition from the configuration with a cluster around the leader to the unification of opinions in the whole group. There is a well defined temperature T_c that separates these two phases. To estimate the dependence of T_c on other system parameters analytically, one can use a mean field approach, like methods developed in [25,27]. The two limiting cases of such an approach correspond to low temperature and high temperature approximations and are discussed in the following.

4.4 Low–temperature mean–field approximation

For *low temperatures* T, i.e. for a small noise level, the cluster of leaders followers is only slightly diluted and its *effective radius* $a(T)$ can be treated as an order parameter. One can then calculate the impact $I(d)$ acting on the group member inside ($d < a$) and outside ($d > a$) the cluster respectively [25]:

$$I_i(d) = -\frac{s_L}{d} - 8a\,E\left(\frac{d}{a},\frac{\pi}{2}\right) + 4R\,E\left(\frac{d}{R},\frac{\pi}{2}\right) + 2\sqrt{\pi} - \beta, \tag{19}$$

$$I_o(d) = \frac{s_L}{d} + 8a\,E\left(\frac{d}{a},\arcsin\frac{a}{d}\right) - 4R\,E\left(\frac{d}{R},\frac{\pi}{2}\right) + 2\sqrt{\pi} - \beta, \tag{20}$$

where $E(k,\varphi) = \int_0^{\varphi}(1 - k^2\sin^2\alpha)^{1/2}d\alpha$ is the elliptic integral of the second kind. Both functions are plotted in Fig. 2 for $s_L = 400$. The system remains in equilibrium, therefore the impact on every individual is negative (nobody changes his/her opinion). It approaches zero at the border of the cluster which means that individuals located in the neighbourhood of that border are most sensitive to thermal fluctuations. We can however observe a significant *asymmetry* of the impact. It is considerably stronger inside the cluster. Individuals near the leader are deeper confirmed in their opinion, so they are also more resistant against noise in dynamics. When we increase the temperature starting from $T \simeq 0$, random opinion changes begin. Primarily it concerns those near the border (the weakest impact). As a result individuals with adverse opinions appear both inside and outside the cluster. They are more numerous outside because of the weaker impact (cf. Fig. 2).

Effectively, we observe the growth of a minority group. This causes the supportive impact outside the cluster to become still weaker and the majority to become more sensitive to random changes. It is a kind of positive feedback. At a certain temperature, the process becomes an avalanche, and the former majority disappears. Thus, noise induces a jump from one attractor (cluster) to another (unification). Such a transition is possible for every non-zero temperature, but its probability remains negligible until the noise level exceeds a certain critical value that corresponds to our critical temperature T_c.

Using Eq. (14) and taking into account Eqs. (19) and (20) we can compute the probability $\Pr(\sigma = 1)(r)$ that an individual at the distance r from the leader, shares opinion $+1$, which is assumed as the opinion of the leader. Then, the mean number of all individuals with opinion $+1$ may be calculated by integrating this probability multiplied by the surface density (equaling 1) over the whole space:

$$\overline{n(\sigma=1)(T)} = \int_0^R \Pr(\sigma=1)(r)\, 2\pi r\, dr. \tag{21}$$

This number equals the effective area of the circular cluster, so its radius is

$$\overline{a(T)} = \sqrt{\frac{\overline{n(\sigma=1)(T)}}{\pi}}. \tag{22}$$

Eq. (22) is a rather involved transcendental equation for $a(T)$ (it appears on the right hand side in $I_i(r)$ and $I_o(r)$). For low temperatures T it has three solutions $a(T)$ corresponding to a stable cluster, an unstable cluster and a socially homogeneous state. The numerical solution for the radius of the stable cluster is presented in Fig. 3 together with results of computer simulations. One should point out that the radius of the cluster a is an increasing function of the temperature T for the reasons discussed above. At some critical temperature, a pair of solutions corresponding to the stable and the unstable cluster coincide [25, 27]. Above this temperature, there exists *only* the solution corresponding to the socially homogeneous state. Fig. 4 shows the plot of the critical temperature T_c obtained from (22) as the function of the leader strength s_L together with results of computer simulations.

4.5 High–temperature mean–field approximation

For high temperatures or small values of the leader's strength s_L, the cluster around the leader is very diluted and it is more appropriate to assume that there is a *spatially homogeneous mixture* of leaders followers and opponents, instead of a *localized cluster* with a radius $a(T)$. It follows that at each site there is *the same* probability $0 < p(T) < 1$ to find an individual sharing the leaders opinion, and $p(T)$ plays the role of order parameter. Neglecting the self–support ($\beta = 0$) one can write the social impact acting on a opponent of the leader at place x as [27]:

$$I(x) = \frac{s_L}{g(x)} + (2p-1)\rho\bar{s}J_D(x) + h \tag{23}$$

$J_D(x) = \int_{D_R} 1/g(|\mathbf{r}-\mathbf{x}|)d^2\mathbf{r}$ is a function which depends only on the size of the group and the type of interactions. After a short algebra one gets the following equation for the probability $p(T)$ [27]:

$$p = \frac{1}{\pi R^2 \rho}\int_0^R \rho \Pr(r)\, 2\pi r\, dr = \frac{1}{R^2}\int_0^R \frac{\exp\left[I(r,p)/T\right]}{\cosh\left[I(r,p)/T\right]} r dr \equiv f(p), \tag{24}$$

where $I(x,p)$ is given by (23). Similar to equation (22) obtained for low temperatures, there are three solutions of Eq. (24): the smallest one corresponds to the stable cluster around the leader, the middle one to the unstable cluster which, in fact, is not observed, and the largest one to the unification. The size of the stable cluster grows with increasing temperature up to a critical value T_c when it coincides with the unstable solution. At this temperature, a transition from a stable cluster to unification occurs [27]. For $T > T_c$, unification is the only solution, but it is no longer a perfect unification because due to the noise individuals of the opposite opinion appear. When the temperature increases further, $p(T)$ tends to 1/2 which means that the dynamics is random and both opinions appear with equal probability.

5 Social temperature as a dynamical variable

So far we treated the social temperature as a system parameter which may change due to some arbitrary reasons, but the changes are not influenced by the changes of opinions (only the opposite is of course true). In order to capture also this aspect in the model we introduce a coupling between the changes of global minority-majority proportion and the social temperature, understood as the degree of randomness in the process of decision making by individuals, or the volatility. We assume that large changes in the opinion distributions increase the temperature because first, many individuals who have just changed their minds are still not deeply convinced to their new opinion and second, large scale changes encourage others to verify and possibly change their opinions. On the other hand, when the global minority-majority ratio remains constant, the temperature decreases due to the general lack of changes (social inertia) or “weariness” by endless changes (if the temperature remains high inducing almost random permanent changes of opinions and thus a constant global ratio).

Based on these assumptions, we suggest the following rule for the dynamics of T:

$$T(t+1) = T_0 + \gamma[T(t) - T_0 + a(\sigma(t) - \sigma(t-1))^2] \tag{25}$$

where $\sigma(t) = \sum_{i=1}^{N} \sigma_i(t)/N$ is the average instantaneous opinion. T_0 is the “background temperature” independent of the changes of opinion, the constant $\gamma < 1$ describes the rate of “cooling down”, while $a > 0$ the coupling between the opinion and temperature changes.

Let us consider the case without the leader. Using the mean field approximation one gets from (13) and (14) the dynamics of σ:

$$\sigma(t+1) = \tanh\left[\frac{\sigma(t)\overline{s}\,\overline{m}}{T(t)}\right] \tag{26}$$

The equations (25) and (26) can be transformed into a three-dimensional map describing the *global* dynamics of the system:

$$\begin{aligned}\sigma(t+1) &= \tanh[\frac{\sigma(t)}{\theta(t)}] \\ \theta(t+1) &= \theta_0 + \gamma[\theta(t) - \theta_0 + c(\sigma(t) - \sigma_p(t))^2] \\ \sigma_p(t+1) &= \sigma(t)\end{aligned} \tag{27}$$

where $\theta = T/(\overline{s}\,\overline{m})$, $c = a/(\overline{s}\,\overline{m})$.

It can be easily checked that the map has a fixed point ($\sigma = 0, \theta = \theta_0$) which is stable for $\theta_0 > 1$. For $\theta_0 < 1$ it becomes unstable and two stable symmetric fixed points (σ^*, θ_0) appear (pitchfork bifurcation), where σ^* are the solutions of the equation $\sigma^* = \tanh(\sigma^*/\theta_0)$. However, at least for some parameter values, a weakly chaotic attractor coexists. The dynamics has the form of oscillations with slightly (chaotically) fluctuating period and amplitude. These oscillations correspond to permanent switching: "high temperature phase"$\sigma \approx 0$ → "cooling down" → "low temperature phase" $\sigma \approx \sigma^*$ → rapid change of σ, "heating up" → "high temperature phase". The example of such dynamics is shown in Fig. 5.

Since the mean field approximation is correct only for large N this kind of chaotic oscillations can be observed in systems with a finite number of individuals only as transients; after some time a convergence to the stable fixed points occurs.

6 Modelling opinion dynamics by means of active Brownian particles

6.1 The model

There are several basic disadvantages of the models considered in the previous chapters. In particular, it is assumed, that the impact on an individual is updated with infinite velocity, and no memory effects are considered. Further, there is no migration of the individuals, and any "spatial" distribution of opinions refers to a "social", but not to the physical space.

An alternative approach [31] to the social impact model of collective opinion formation, which tries to include these features is based on *active Brownian particles* [28–30, 32, 33], which interact via a *communication field*. This scalar field considers the spatial distribution of the individual opinions, further, it has a certain life time, reflecting a collective memory effect and it can spread out in the community, modeling the transfer of information.

The spatio-temporal change of the communication field is given by the following equation:

$$\frac{\partial}{\partial t} h_\sigma(\boldsymbol{r}, t) = \sum_{i=1}^{N} s_i \delta_{\sigma,\sigma_i}\ \delta(\boldsymbol{r} - \boldsymbol{r}_i)\ - \gamma h_\sigma(\boldsymbol{r}, t) + D_h \Delta h_\sigma(\boldsymbol{r}, t). \tag{28}$$

Every individual contributes permanently to the field $h_\sigma(\boldsymbol{r},t)$ with its opinion σ_i and with its personal strength s_i at its current spatial location $\boldsymbol{r}_i$. Here, δ_{σ,σ_i} is the Kronecker Delta, $\delta(\boldsymbol{r}-\boldsymbol{r}_i)$ denotes Dirac's Delta function used for continuous variables, N is the number of individuals. The information generated by the individuals has a certain average life time $1/\gamma$, further it can spread throughout the system by a diffusion-like process, where D_h represents the diffusion constant for information exchange. If two different opinions are taken into account, the communication field should also consist of two components, $\sigma = \{-1, +1\}$, each representing one opinion.

In this model, the scalar *spatio-temporal communication field* $h_\sigma(\boldsymbol{r},t)$ [31], plays in part the role of social impact I_i used in [25, 27]. Instead of a social impact, the communication field $h_\sigma(\boldsymbol{r},t)$ influences the individual i as follows: At a certain location $\boldsymbol{r}_i$, the individual with opinion $\sigma_i = +1$ is affected by two kinds of information: the information resulting from individuals who share his/her opinion, $h_{\sigma=+1}(\boldsymbol{r}_i,t)$, and the information resulting from the opponents $h_{\sigma=-1}(\boldsymbol{r}_i,t)$. Dependent on the *local* information, the individual reacts in two ways: (i) it can *change its opinion*, (ii) it can *migrate* towards locations which provide a larger support of its current opinion. These opportunities are specified in the following.

We assume that the probability $p_i(\sigma_i,t)$ to find the individual i with the opinion σ_i changes in the course of time due to the master equation (the dynamics is continuous in time):

$$\frac{d}{dt}p_i(\sigma_i,t) = \sum_{\sigma_i'} w(\sigma_i|\sigma_i')p_i(\sigma_i',t) - p_i(\sigma_i,t)\sum_{\sigma_i'} w(\sigma_i'|\sigma_i). \tag{29}$$

where the transition rates are described similar to Eq. (14)

$$w(\sigma_i'|\sigma_i) = \eta \exp\{[h_{\sigma'}(\boldsymbol{r}_i,t) - h_\sigma(\boldsymbol{r}_i,t)]/T\} \quad \text{for} \quad \sigma \neq \sigma' \tag{30}$$

and $w(\sigma_i|\sigma_i) = 0$. The movement of the individual located at space coordinate $\boldsymbol{r}_i$ is described by the following overdamped Langevin equation:

$$\frac{d\boldsymbol{r}_i}{dt} = \alpha_i \left.\frac{\partial h_e(\boldsymbol{r},t)}{\partial r}\right|_{\boldsymbol{r}_i} + \sqrt{2\,D_n}\,\xi_i(t). \tag{31}$$

In the last term of Eq. (31) D_n means the spatial diffusion coefficient of the individuals. The random influences on the movement are modeled by a stochastic force with a δ-correlated time dependence, i.e. $\xi(t)$ is white noise with $\langle \xi_i(t)\,\xi_j(t')\rangle = \delta_{ij}\,\delta(t-t')$. The term $h_e(\boldsymbol{r},t)$ in Eq. (31) means an *effective* communication field which results from $h_\sigma(\boldsymbol{r},t)$ as a certain function of both components, $h_{\pm1}(\boldsymbol{r},t)$ [31]. The parameters α_i are individual response parameters. In the following, we will assume $\alpha_i = \alpha$ and $h_e = h_\sigma$.

6.2 Critical conditions for spatial opinion separation

The spatio-temporal density of individuals with opinion σ can be obtained as follows:

$$n_\sigma(\boldsymbol{r},t) = \int \sum_{i=1}^{N} \delta_{\sigma,\sigma_i}\delta(\boldsymbol{r}-\boldsymbol{r}_i)P(\sigma_1,\boldsymbol{r}_1...,\sigma_N,\boldsymbol{r}_N,t)d\boldsymbol{r}_1...d\boldsymbol{r}_N \tag{32}$$

$P(\underline{\sigma},\underline{r},t) = P(\sigma_1,\boldsymbol{r}_1,...,\sigma_N,\boldsymbol{r}_N,t)$ is the canonical N-particle distribution function which gives the probability to find the N individuals with the opinions $\sigma_1,...,\sigma_N$ in the vicinity of $\boldsymbol{r}_1,....,\boldsymbol{r}_N$ on the surface A at time t. The evolution of $P(\underline{\sigma},\underline{r},t)$ can be described by a master equation [31] which considers both Eqs. (30), (31). Neglecting higher order correlations, one obtains from Eq. (32) the following reaction-diffusion equation for $n_\sigma(\boldsymbol{r},t)$ [29, 31]:

$$\begin{aligned}\frac{\partial}{\partial t}\, n_\sigma(\boldsymbol{r},t) = &- \nabla\left[n_\sigma(\boldsymbol{r},t)\,\alpha\,\nabla h_\sigma(\boldsymbol{r},t)\right] + D_n\,\Delta n_\sigma(\boldsymbol{r},t)\\ &- \sum_{\underline{\sigma'\neq\sigma}}\left[w(\sigma'|\sigma)\,n_\sigma(\boldsymbol{r},t) + w(\sigma|\sigma')\,n_{\sigma'}(\boldsymbol{r},t)\right]\end{aligned} \tag{33}$$

with the transition rates given by eq. (30). Eq. 33 together with Eq. 28 form a set of four equations describing our system completely.

Now, let us assume that the spatio-temporal communication field *relaxes faster* than the related distribution of individuals to a quasi-stationary equilibrium. The field $h_\sigma(\boldsymbol{r},t)$ should still depend on time and space coordinates, but, due to the fast relaxation, there is a fixed relation to the spatio-temporal distribution of individuals. Further, we neglect the independent diffusion of information, assuming that the spreading of opinions is due to the migration of the individuals. Using $\dot{h}_\sigma(\boldsymbol{r},t) = 0$, $s_i = s$ and $D_h = 0$ we get:

$$h_\sigma(\boldsymbol{r},t) = \frac{s}{\gamma}\,n_\sigma(\boldsymbol{r},t) \tag{34}$$

Inserting Eq. (34) into Eq. (33) we reduce the set of coupled equations to two equations.

The homogeneous solution for $n_\sigma(\boldsymbol{r},t)$ is given by the mean densities:

$$\bar{n}_\sigma = \frac{\bar{n}}{2} \qquad \text{where} \qquad \bar{n} = \frac{N}{A} \tag{35}$$

Under certain conditions however, the homogeneous state becomes unstable and a spatial separation of opinions occurs. In order to investigate these critical conditions, we allow small fluctuations $\delta n_\sigma \sim \exp(\lambda t + i\boldsymbol{k}\boldsymbol{r})$ around the homogeneous state $\bar{n}_\sigma$ and perform a linear stability analysis [31]. The resulting dispersion relations read:

$$\begin{aligned}&\lambda_1(\boldsymbol{k}) = -k^2\,C + 2B\ ;\ \ \lambda_2(\boldsymbol{k}) = -k^2\,C\\ &B = \frac{\eta\,s\,\bar{n}}{\gamma T} - \eta\ ;\ \ C = D_n - \frac{\alpha s\,\bar{n}}{2\gamma}\end{aligned} \tag{36}$$

It follows that stability conditions of the homogeneous state, $n_\sigma(\boldsymbol{r}, t) = \bar{n}/2$, can be expressed as:

$$T > T_1^c = \frac{s\,\bar{n}}{\gamma} \quad ; \quad D > D_n^c = \frac{\alpha}{2}\,\frac{s\,\bar{n}}{\gamma} \tag{37}$$

If the above conditions are not fulfilled, the homogeneous state that corresponds to a *paramagnetic phase* is unstable (i) against the formation of spatial "domains" where one of opinions $\sigma = \pm 1$ *locally dominates*, or (ii) against the formation of a *ferromagnetic* state where the *total number* of people sharing both opinions are not equal.

Case (i) can occur only for a systems whose linear dimensions are large enough, so that large–scale fluctuations with small wave numbers can destroy the homogeneous state [31]. In case (ii), each subpopulation can exist either as a *majority* or as a *minority* within the community. Which of these two possible situations is realized, depends in a deterministic approach on the initial fraction of the subpopulation. Breaking the symmetry between the two opinions due to *external influences* (support) for one of the opinions would increase the region of initial conditions which lead to a majority status. Above a critical value of such a support, the possibility of a minority status completely vanishes and the supported subpopulation will grow towards a majority, regardless of its initial population size, with no chance for the opposite opinion to be established [31].

7 Conclusions

The Nowak-Szamrej-Latané models can be described within a mean field theory, using in some cases quite complicated order parameters. The presence of the social clusters and "staircase dynamics" are the generic properties of the models. Adding learning mechanism (changes of individual social strengths) to the opinion dynamics can induce the emergence of leaders in social groups. In the presence of a strong leader situated in the centre of a finite group, a transition can take place from a state with a cluster around the leader to a state of uniform opinion distribution where virtually all members of the group share the leaders's opinion. The transition occurs if a leader's strength exceeds a well defined critical value or if the noise level ("social temperature") is high enough. The weaker the leader's strength is, the larger is the needed noise. The value of the critical temperature can be calculated using mean field methods where either the existence of an effective value of the cluster radius (low temperature method) or a spatially homogeneous mixture of both opinions (high temperature method) is assumed. Numerical simulations confirm the analytic results. When the social temperature is coupled to the mean opinion dynamics (in a mean field approximation) chaotic oscillations of both quantities can appear.

The extension of the social impact model can be based on the concept of active Brownian particles which communicate via a scalar, multi-component communication field. This allows us to take into account effects of spatial migration (drift and diffusion), a finite velocity of information exchange and memory effects. We have obtained a reaction-diffusion equation for the density of individuals with a certain opinion. In this model, the transition can take place between the "paramagnetic" phase, where the probability to find any of opposite opinions is 1/2 at each place (a high temperature and a high diffusion phase), the "ferromagnetic" phase with a global majority of one opinion and a phase with spatially separated "domains" with a local majority of one opinion.

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Figures

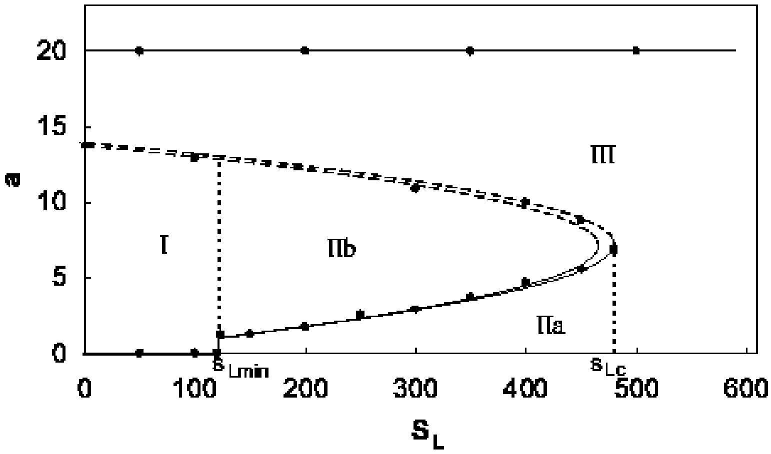

Figure 1: Cluster's radius a vs. leader's strength s_L – phase diagram for circular social space. Interactions proportional to inverse of mutual distance ($I \propto 1/r$). Lines correspond to analytical results, points to computer simulations.

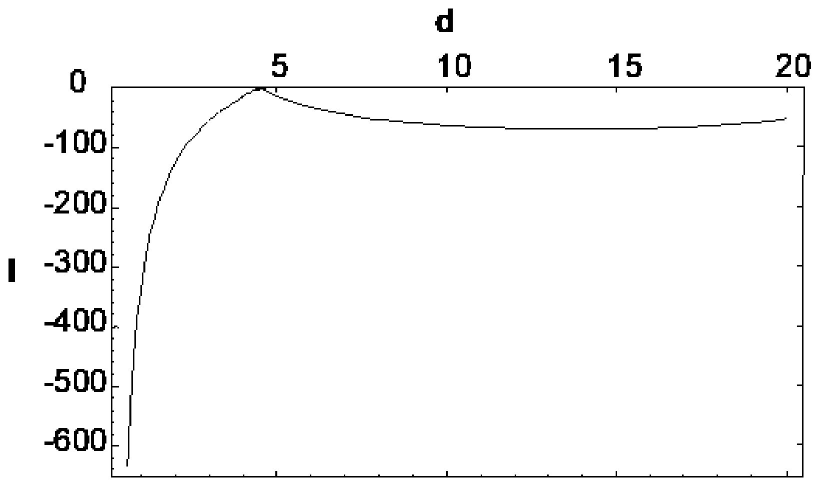

Figure 2: Social impact I as a function of distance d to the leader. Leader's strength $s_L = 400$.

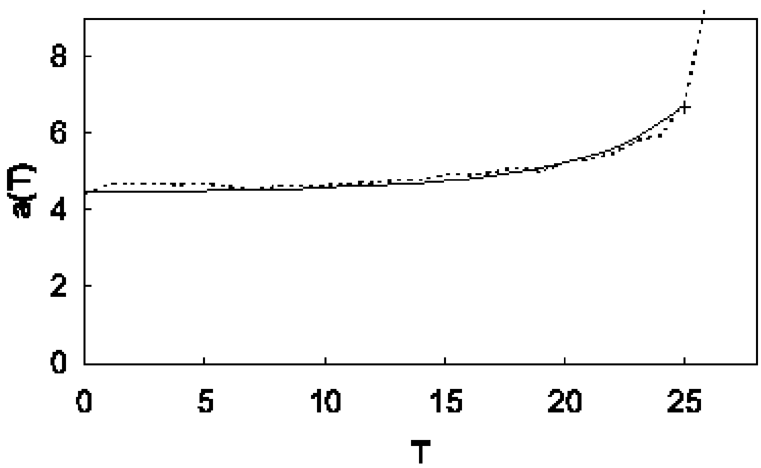

Figure 3: Mean cluster radius a vs. temperature T; $s_L = 400$. Results of calculation (solid) and computer simulation (dotted).

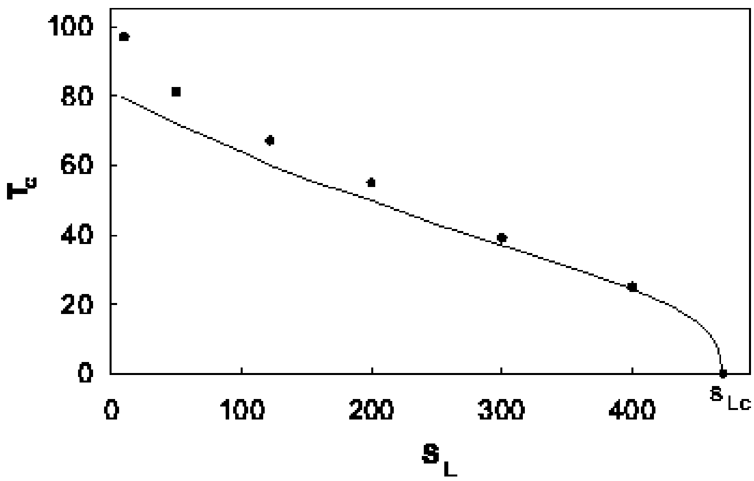

Figure 4: Critical temperature T_c (above which no stable cluster exists) vs. leader's strength s_L. Leader's opinion fixed (independent of the group). Line – calculations (Eq. (22)), points – simulations.

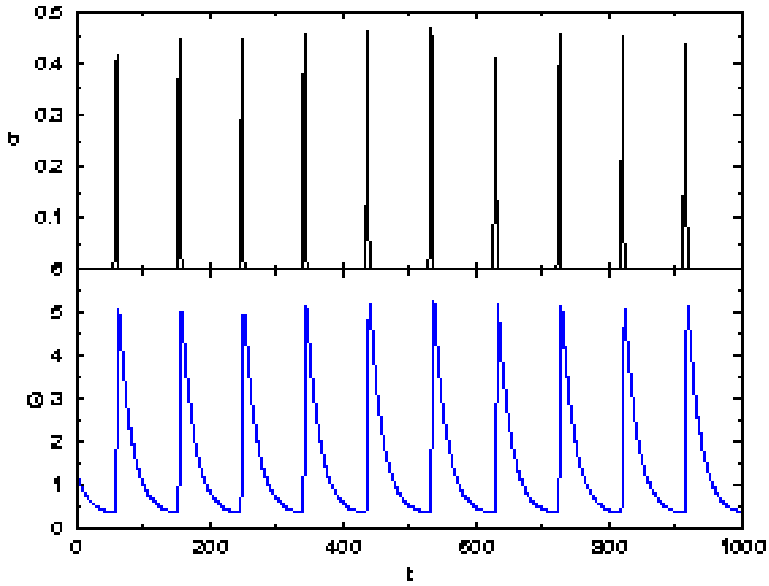

Figure 5: Time evolution of the map (27) for $\theta_0 = 0.3, b = 0.95, c = 40$.