Stochastic Growth Models

- so far: macroscopic/aggregated dynamics, no fluctuations
- now: microscopic view ⇒ single firms, stochastic influences
- modeling aims:
  - reproduce observed power-law distributions
  - derive macroscopic growth dynamics
  - allow fitting of real data

Gibrat's Model

- $x_i(t)$: size of firm i at time t
- "Law of proportionate growth" (Gibrat '30, '31; Sutton '97)

$$x_i(t+\Delta t) - x_i(t) = b_i(t)\, x_i(t)$$

- Assumptions:
  - $b_i(t)$: independent of i, no temporal correlations (random noise)
  - no interactions between firms: $x_i \Rightarrow x$
  - $t \gg \Delta t$:

$$x(t) = x(0)\big(1+b(1)\big)\big(1+b(2)\big)\cdots\big(1+b(t)\big)$$

  - *growth rates:* $R(t) = x(t)/x(0)$, $\quad \ln(1+b) \approx b$

$$\ln R(t) = \sum_{n=1}^{t} b(n)$$

$\Rightarrow$ random walk!! $R(t)$: log-normal distribution

Log-normal vs Power Law Distribution

$$x_{t+1} = \lambda_t \, x_t \quad \text{with } \lambda = b + 1$$

$$P(x_t) = \frac{1}{\sqrt{2\pi Dt}} \frac{1}{x_t} \exp\left[-\frac{1}{2Dt}(\log x_t - vt)^2\right]$$

$$v = \langle \log \lambda \rangle \; ; \quad D = \langle (\log \lambda)^2 \rangle - \langle \log \lambda \rangle^2$$

rewriting:

$$P(x_t) = \frac{1}{\sqrt{2\pi Dt}} \frac{1}{x_t^{1+\mu(x_t)}} e^{\mu(x_t)vt} \; ; \quad \mu(x_t) = \frac{1}{2Dt} \log \frac{x_t}{e^{vt}}$$

$\mu(x_t)$: slowly varying function of x_t

- $x_t \ll e^{(v+2D)t}$ yields $\mu(x_t) \ll 1$
  $\Rightarrow$ log-normal and $1/x_t$ undistuingishable (Montroll & Shlesinger '82)
- however in the tail $x_t \gg e^{(v+2D)t} \Rightarrow \mu(x_t) \to \infty$ (!!)

Drawbacks:

- conceptual:
  - growth/shrinkage not in randomly sized steps
  - firm's dynamics not independent
    profit maximizer $\Rightarrow$ respond similarly to changing market conditions
- empirical:
  - $b_i(t)$ also depend on x
    fluctuations *decrease* as firm's size *increases*
    $\sigma(x) \sim x^{-\beta}$ with $\beta < 0.5$ (Stanley et al. '96, '97)
  - power law instead of log-normal distribution

So, is Gibrat wrong???

Improvements of Gibrat's Model

- economic idea: simple entry dynamics (Simon & Bonini '58)
- mathematic idea: *add more noise!* (Kesten '73)

$$x(t+1) = \lambda(t)\, x(t) + f(t)$$

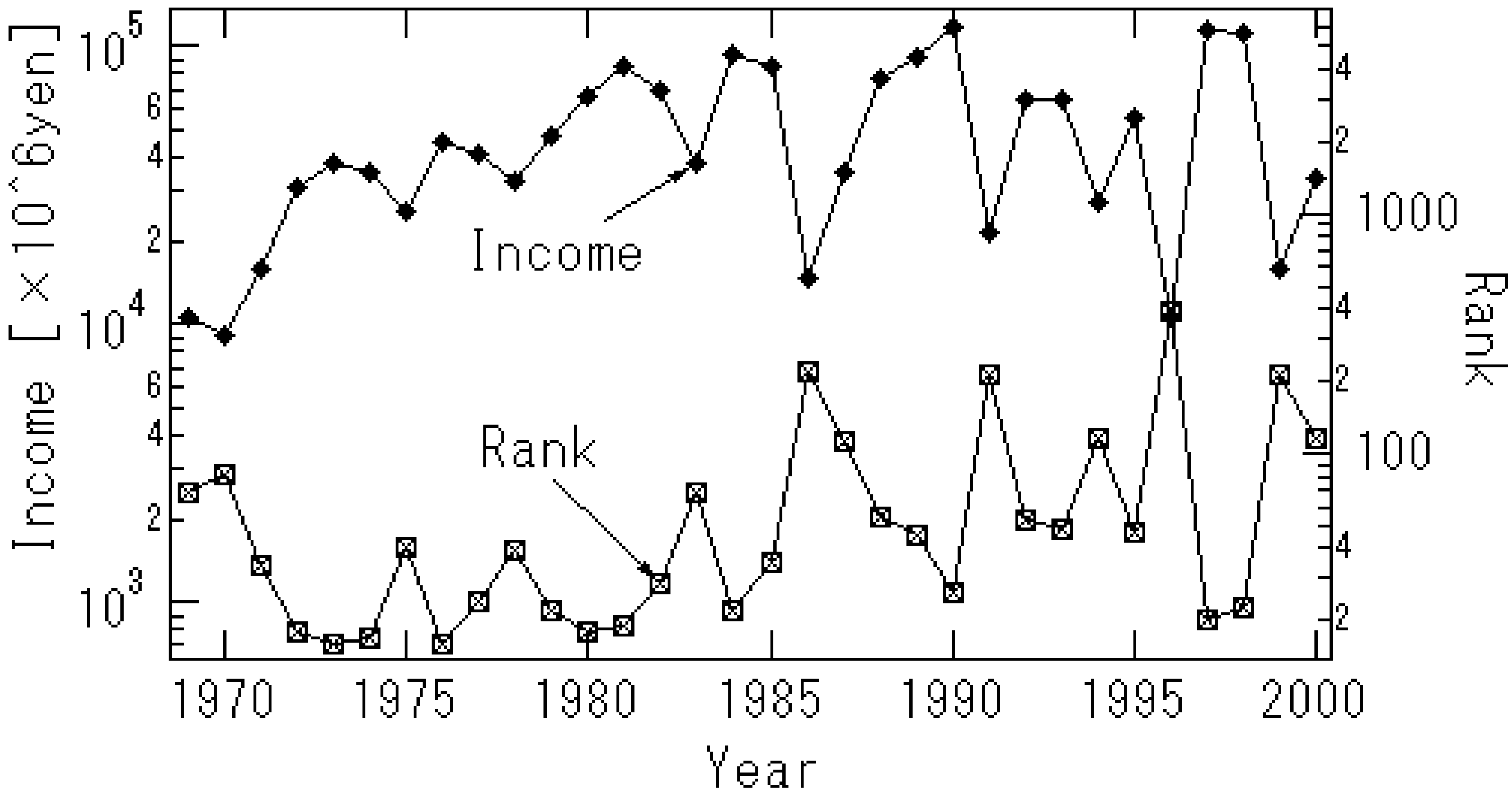
  - λ, f positive indepentent random variables
  - negative drift: $\langle \lambda(t) \rangle < 0$, so $x(t) \to 0$ for $f(t) = 0$
    but for positive random $f(t)$

$$P(x_t) = {x_t}^{-(1+\mu)} \; ; \quad \langle \lambda^{\mu} \rangle = 1$$

- $f(t)$ acts as an "effective repulsion" from zero
  generalization: Sornette & Cont '97
- important: distribution $\varrho(\lambda)$
  the smaller μ, the wilder the fluctuations

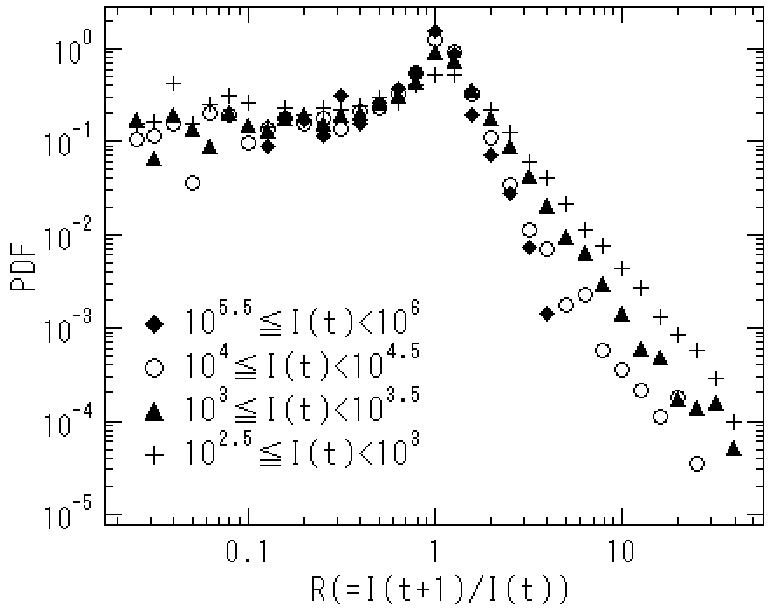
Comparison with real company data

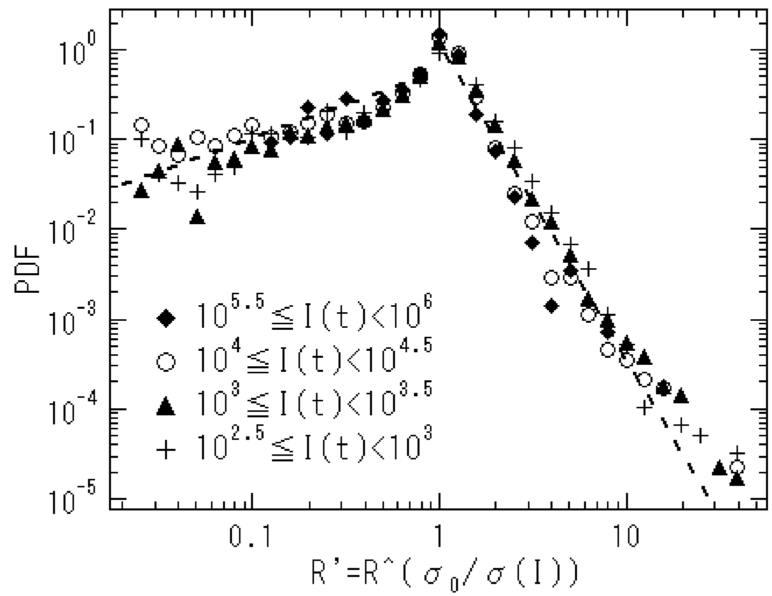
- Takayasu et al ’03: income of 15.000 US and 15.000 non-US comp., 80.000 Japanese comp. (income > 40 Mio Yen), before tax



$$x(t+1) = \alpha(t)\lambda(t,x)\,x(t) + f(t)$$

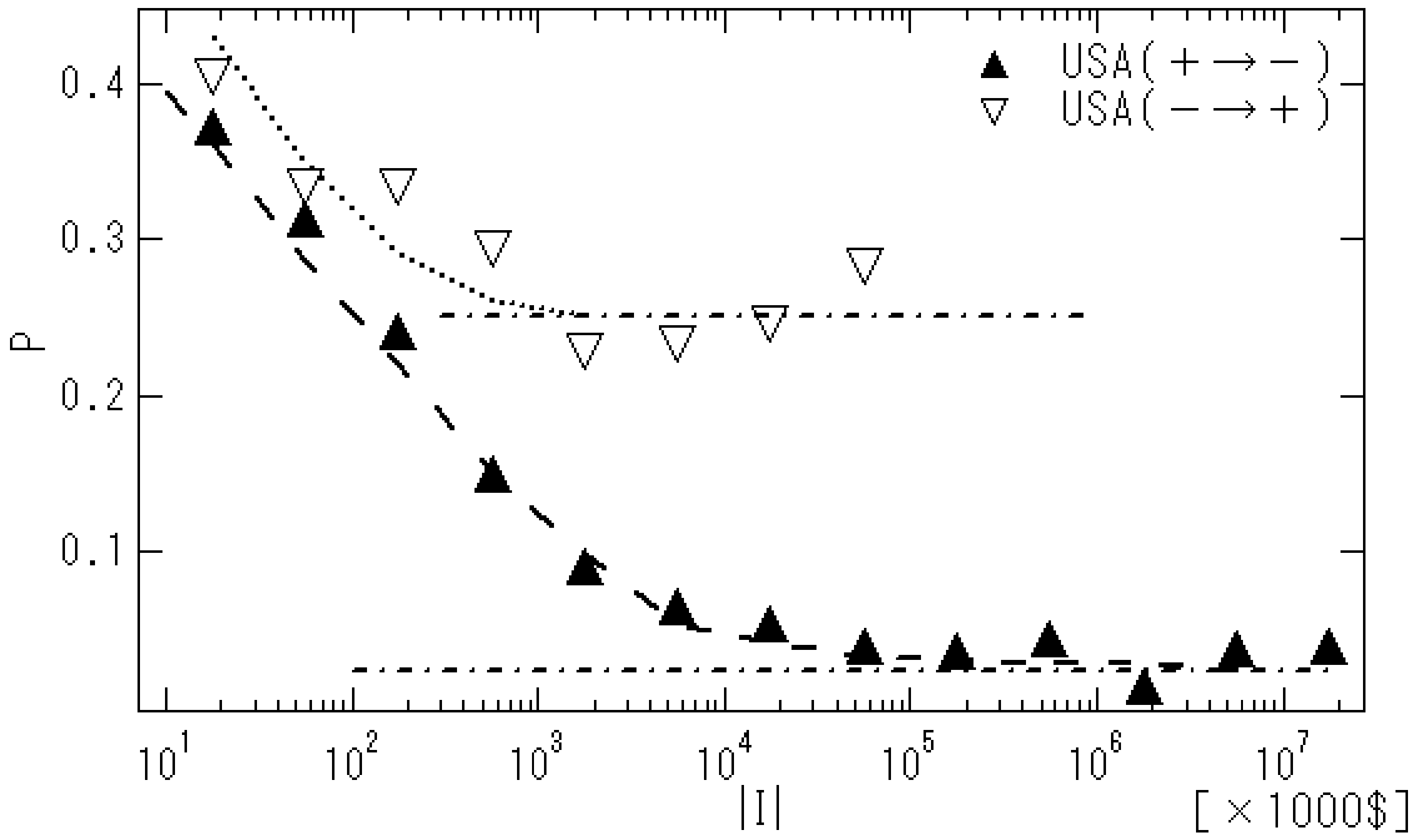
- $\lambda(x,t)$: growth depends on size
  - estimation from $\log$ growth rate: $\log R(t) = \log x(t+1) - \log x(t)$ with standard deviation $\sigma(x)$
    for large x: σ_0, $f(t)/x$ negligible
    scaling by means of normalized growth: $R^{\sigma(x)/\sigma_0}$





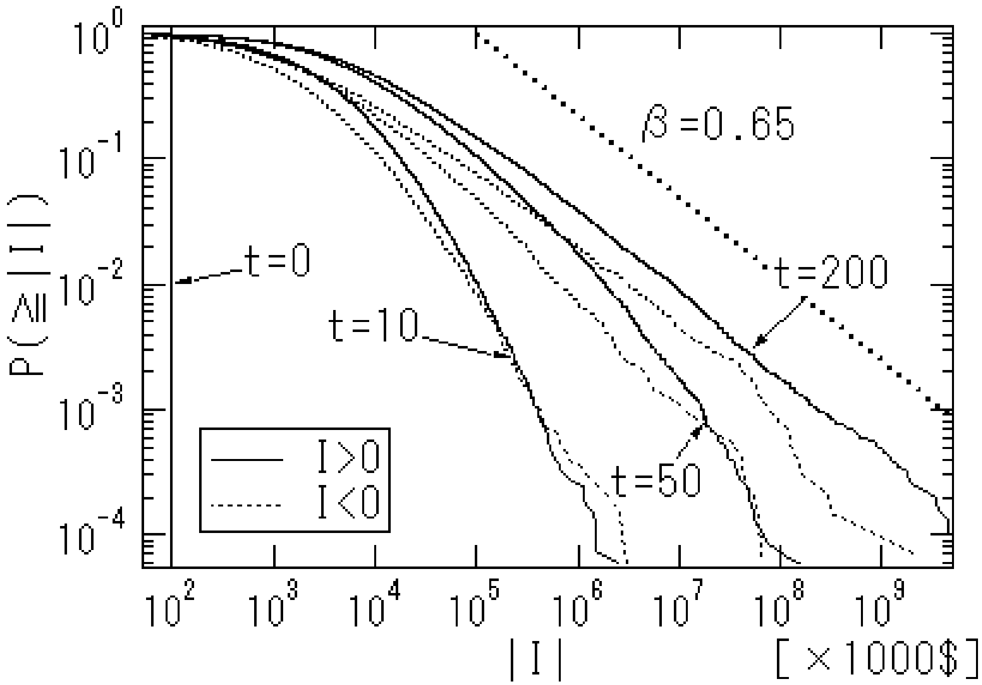
- $\alpha(t)$ either +1 (growth) or (-1) (slump)
  prob. determined empirically from large $|x(t)|$

$$\alpha(t) = \begin{cases} 1 & \text{with prob. } 0.97\ (x(t) > 0),\quad 0.75\ (x(t) < 0) \\ -1 & \text{with prob. } 0.03\ (x(t) > 0),\quad 0.25\ (x(t) < 0) \end{cases}$$



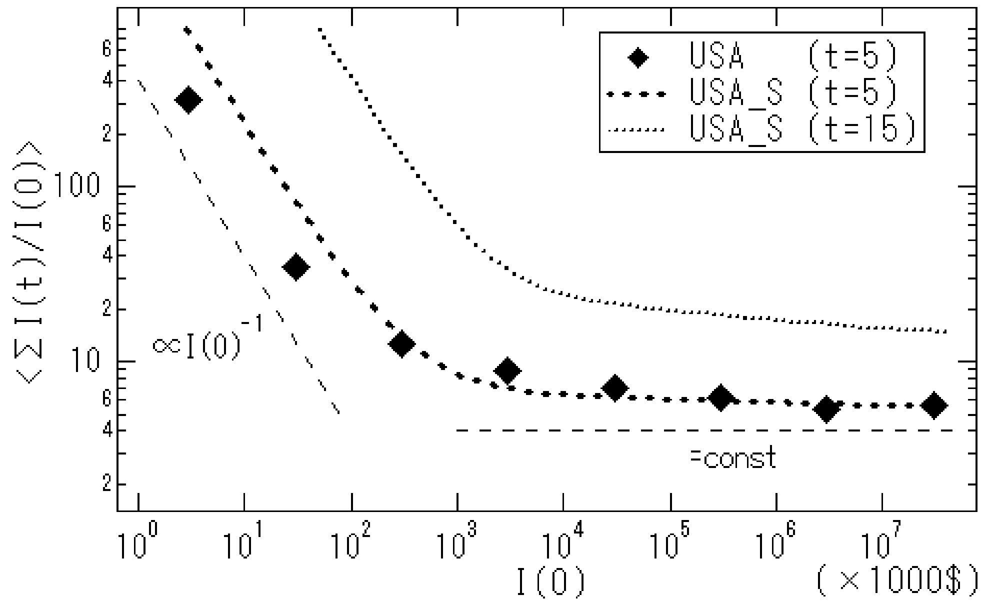
Forecast by means of Monte Carlo Simulations

- initial state: 6.000 companies, $x_i(0) = 100$
  coefficients estimated from real data
- $t = 50$: qualitative agreement with real distribution (US)
  with constant growth rate distribution: firms income will keep growing for more than 100 years



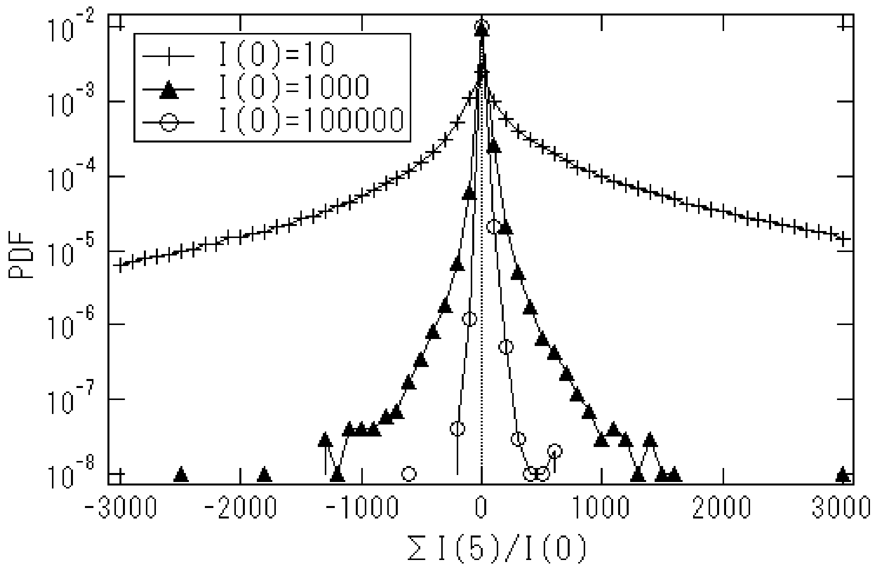
Investment Strategies?

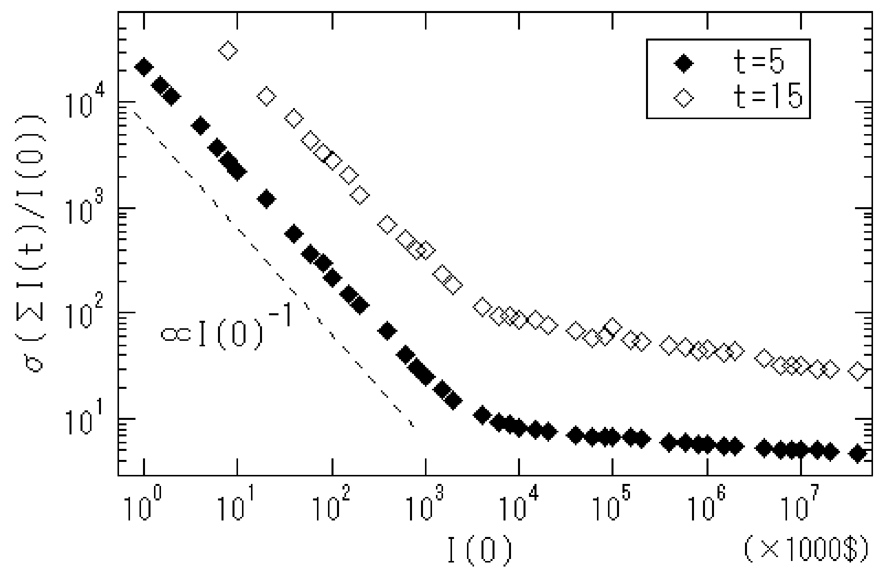
➤ normalized cummulative income for 5 years: $I = \sum_{n=1}^{5} x(n)/x(0)$



➤ for $x(0) > 10^6$\$: $I \propto x(0) \Rightarrow$ invest in small firms?

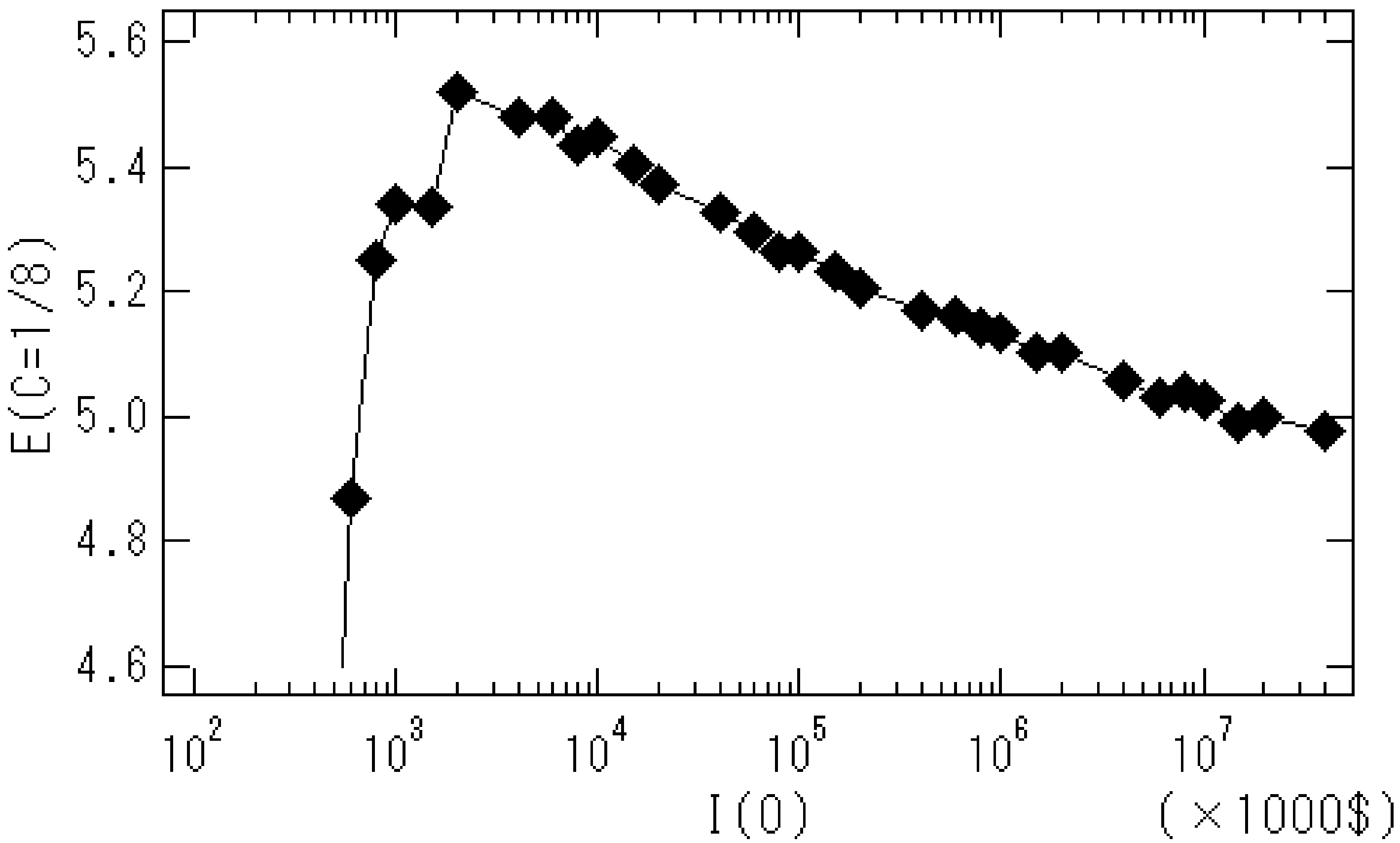
- small firms: large growth rates, but also large variances (notice the asymmetric distribution)





- investment strategy: tradeoff between profits and risks
  investment efficiency: relation between $\langle I \rangle$ and $\sigma(I)$

$$E\big(c,x(0)\big) = \left\langle \sum \frac{x(5)}{x(0)} \right\rangle - c\,\sigma\left(\sum \frac{x(5)}{x(0)}\right)$$



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