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Fraunhofer Institut Autonome Intelligente Systeme

Nichtlineare Votermodelle: Simulation vs. analytische Resultate

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Non-linear Voter Models

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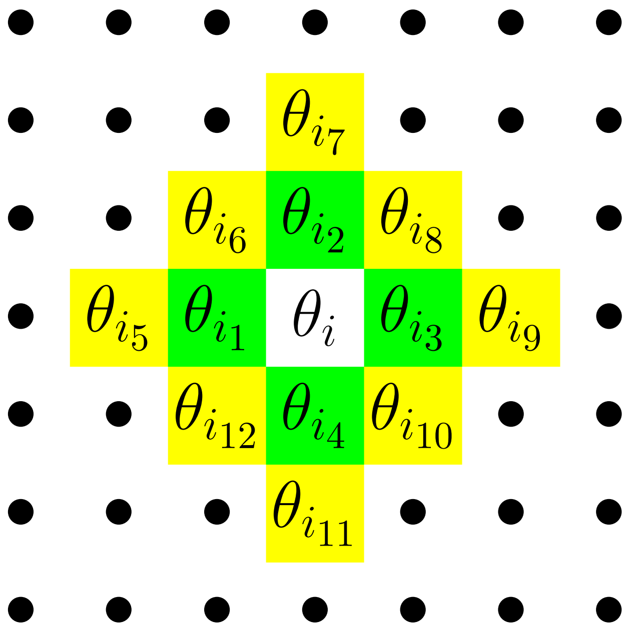
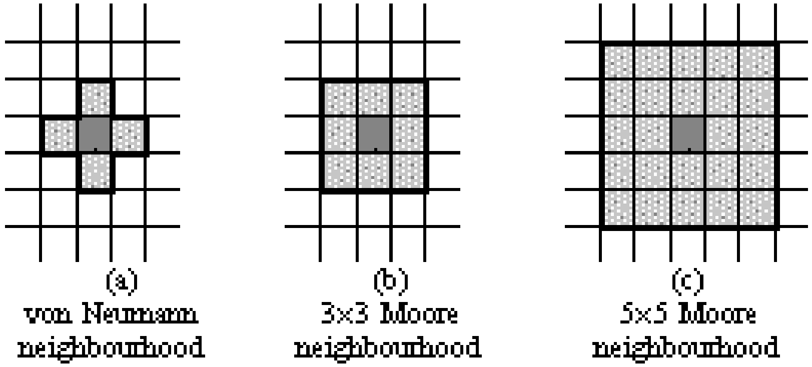
- interdisciplinary enterprise:
  linear voter model: domain of mathematical investigations
  relation to population biology, ecology
- our interest: investigation of spatial effects
  derivation of macro-dynamics from microscopic interactions

Cellular Automaton



- cell i with different states θ_i
- interaction with neighbors j



History: v. Neumann, Ulam (1940s), Conway (1970), Wolfram (1984), ...

Socio/Economy: Sakoda (1949/1971), Schelling (1969), Albin (1975), ...

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  spatial correlation $c_{1|1}$
- stochastic description: $p_i(\theta_i, t) = \sum_{\underline{\theta}_i'} p(\theta_i, \underline{\theta}_i', t)$,
  local neighborhood: $\underline{\theta}_i = \{\theta_{i_1}, \theta_{i_2}, ..., \theta_{i_{n-1}}\}$

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$$\frac{d}{dt}p_i(\theta_i,t) = \sum_{\underline{\theta}_i'} \Big[w(\theta_i|(1-\theta_i),\underline{\theta}_i')\; p(1-\theta_i,\underline{\theta}_i',t) - w(1-\theta_i|\theta_i,\underline{\theta}_i')\; p(\theta_i,\underline{\theta}_i',t) \Big]$$

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- solution:
  (1) stochastic computer simulations
  (2) analytical methods

Local Interaction Rules

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➤ "frequency dependent process": $\underline{\theta}_i \Rightarrow$ local frequency:

$$z_i^{\sigma} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\sigma \theta_{i_j}} \; ; \quad z_i^{(1-\sigma)} = 1 - z_i^{\sigma} \; ; \quad \sigma \in \{0, 1\}$$

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- asymmetric rules: (“Game of Life”, n=9)
  “alive”: $\theta_i = 1 \Rightarrow$ rule set 1: “alive” if 2 or 3 neighbors alive
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- symmetric rules: same for $\theta_i \in \{0, 1\}$

z_i^{σ}	$z_i^{(1-\sigma)}$	$w(1-\theta_i \vert \theta_i=\sigma, z_i^{\sigma})$
1	0	ε
4/5	1/5	α_1
3/5	2/5	α_2
2/5	3/5	$\alpha_3 = 1-\alpha_2$
1/5	4/5	$\alpha_4 = 1-\alpha_1$

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- positive dependence: $0 \le \alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4 \le 1$
  "majority voting" (frequent opinions survive)
- symmetry between opinions: $\alpha_3 = 1-\alpha_2$ and $\alpha_4 = 1-\alpha_1$
- *linear* voter model: $\alpha \propto z_i^{(1-\sigma)}$
  i.e. $\varepsilon = 0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$

- negative dependence: $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq 0$
  "minority voting" (rare opinions survive)

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  "minority voting" (rare opinions survive)
- "allee effects": $\alpha_1 \leq \alpha_2$, $\alpha_2 \geq \alpha_3$, $\alpha_3 \leq \alpha_4$, *etc.*
  voting against the trend

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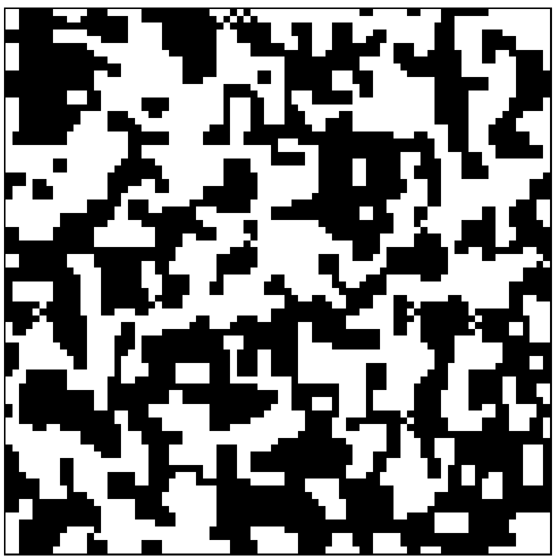
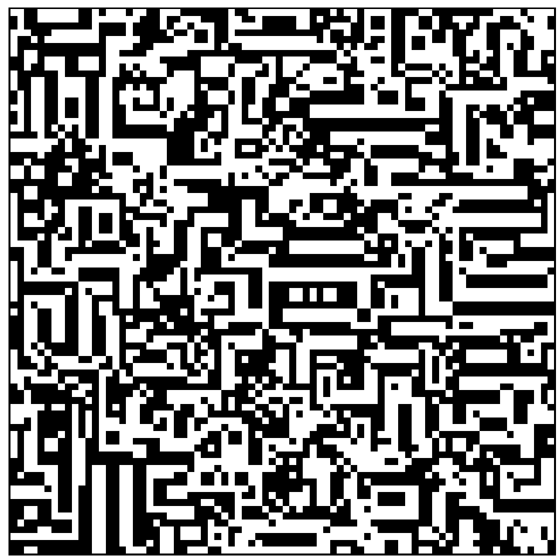
$\varepsilon = 0, \alpha_1 = 0, \alpha_2 = 0$ $\varepsilon = 0, \alpha_1 = 1, \alpha_2 = 1$



$t = 10^2$



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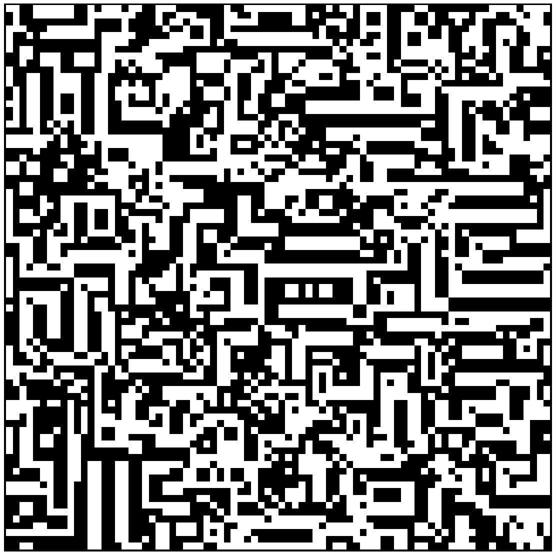
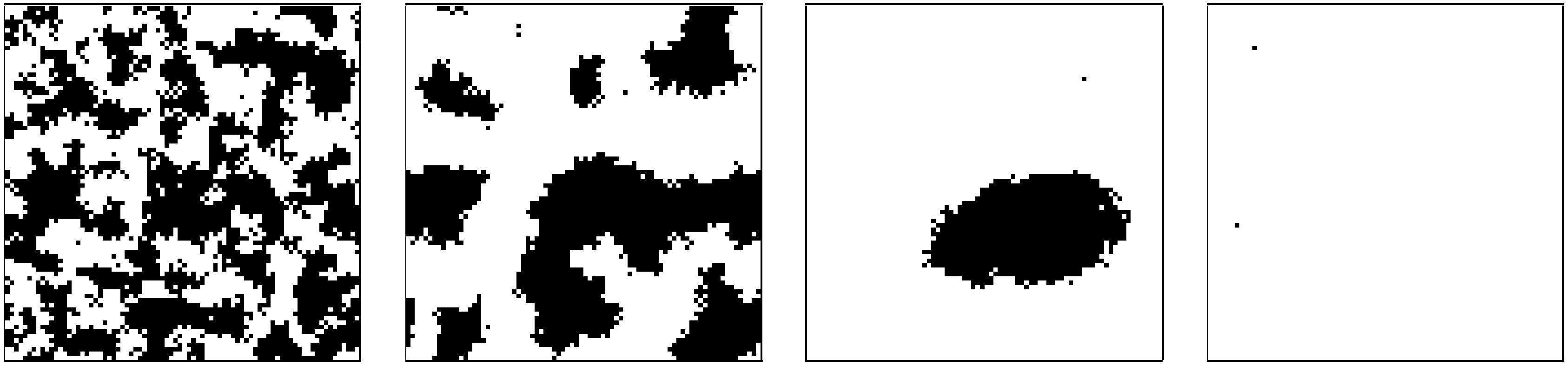
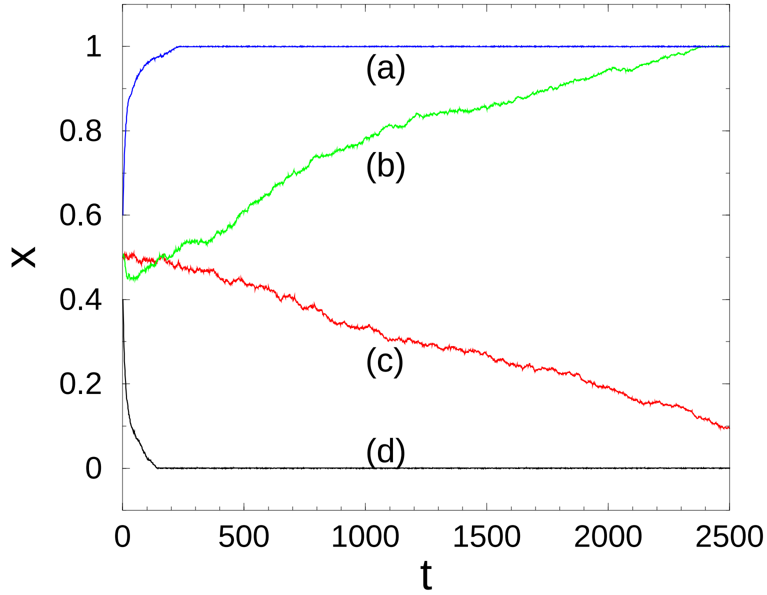


- spatial coexistence of both opinions: $x^{\text{stat}} = 0.5$, "aggregation" of opinions

Stochastic CA

$\varepsilon = 10^{-4}$, $\alpha_1 = 0.1$, $\alpha_2 = 0.3$ $t = 10^1, 10^2, 10^3, 10^4$





Coexistence?

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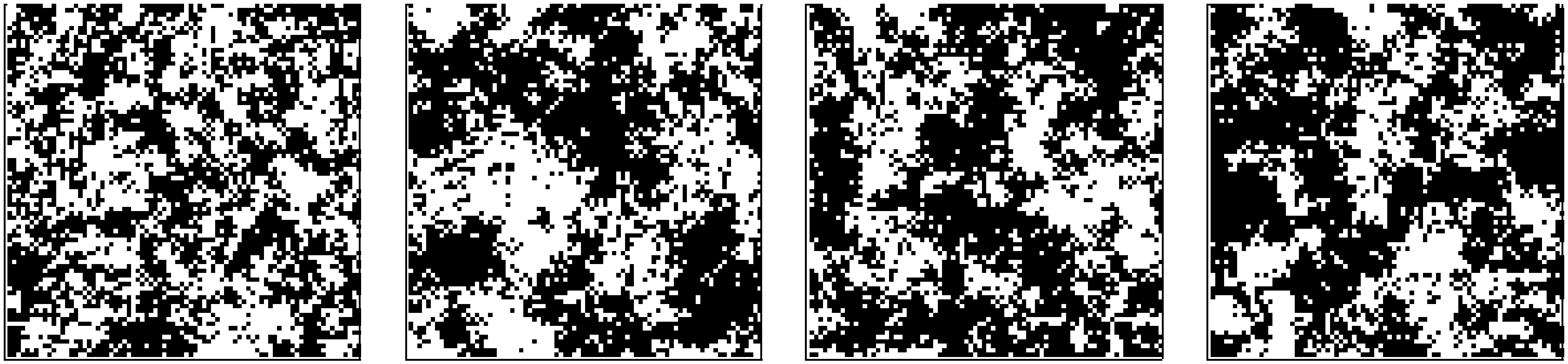
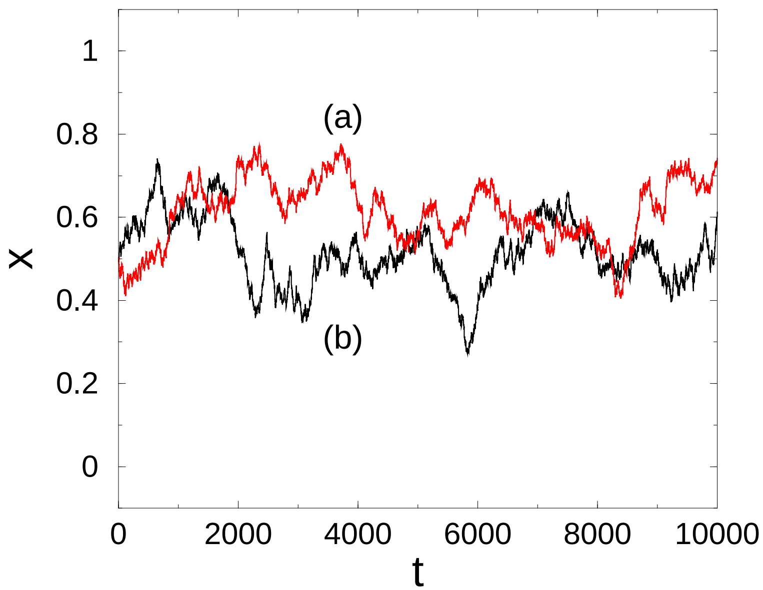
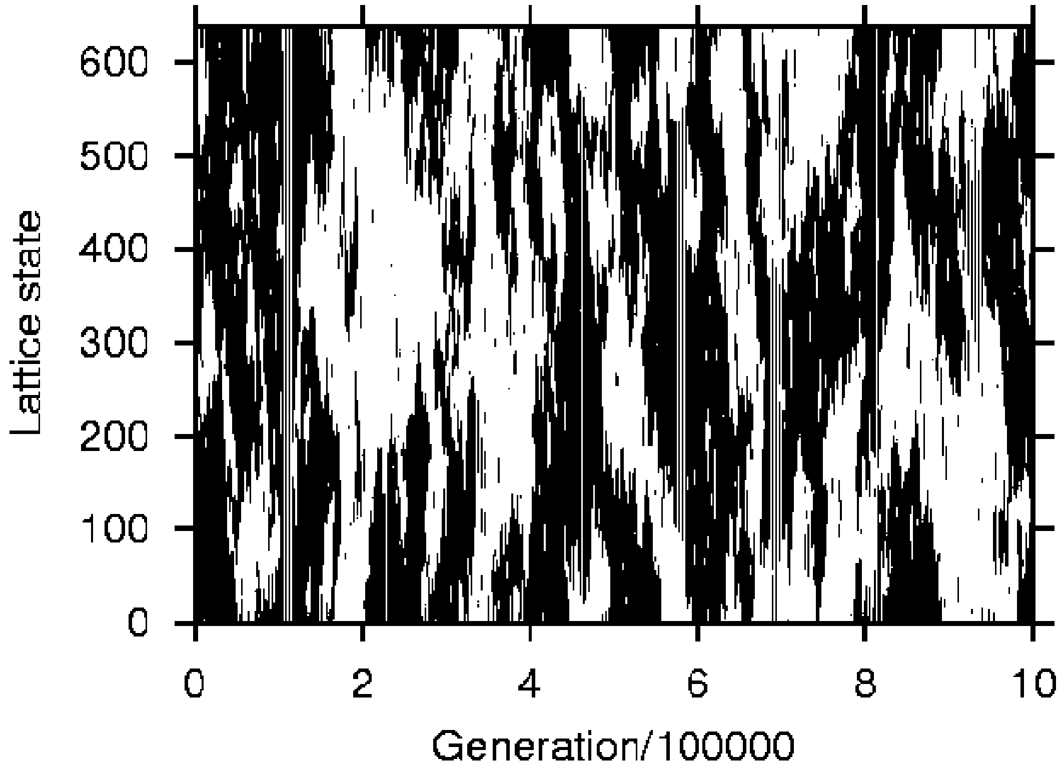
$\varepsilon = 10^{-4}$, $\alpha_1 = 0.25$, $\alpha_2 = 0.25$ $t = 10^1, 10^2, 10^3, 10^4$





(a) $a_1 = 0.2$, $\alpha_2 = 0.4$
(voter model)
(b) $\alpha_1 = 0.25$, $\alpha_2 = 0.25$

1d CA:



long-term nonstationarity; temporal domination of one opinion

Two tasks:

- 1. define range of parameters for coexistence
- 2. describe spatial correlations between opinions

Macroscopic Equations

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- macroscopic variable: $\langle x(t)\rangle = \dfrac{1}{N}\sum_{i=1}^{N} p_i(\theta_i = 1, t)$

$$\frac{d}{dt}\langle x(t)\rangle = \sum_{\underline{\sigma}'} \left[w(1|0,\underline{\sigma}')\ \left\langle x_{0,\underline{\sigma}'}(t)\right\rangle - w(0|1,\underline{\sigma}')\ \left\langle x_{1,\underline{\sigma}'}(t)\right\rangle \right]$$

calculation of $\left\langle x_{\sigma,\underline{\sigma}'}(t)\right\rangle$: consideration of *all* possible $\underline{\sigma}'$ (!)

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- mean-field limit: no spatial correlations

$$\big\langle x_{\underline{\sigma}^0}\big\rangle = \langle x_\sigma\rangle \prod_{j=1}^{m}\big\langle x_{\sigma_j}\big\rangle$$

- mean-field dynamics:

$$\frac{dx}{dt} = \varepsilon\left[(1-x)^5 - x^5\right] + x^5 - x + x(1-x)^4\,(5\alpha_1) + x^2(1-x)^3\,(10\alpha_2) + x^3(1-x)^2\,[10\,(1-\alpha_2)] + x^4(1-x)\,[5\,(1-\alpha_1)]$$

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- stationary solutions: $\dot{x} = 0$, $\varepsilon = 0$

$$x^{(1)} = 0\;;\quad x^{(2)} = 1\;;\quad x^{(3)} = 0.5$$

$$x^{(4,5)} = 0.5 \pm \sqrt{\frac{10\alpha_2 + 15\alpha_1 - 7}{40\alpha_2 - 20\alpha_1 - 12}}$$

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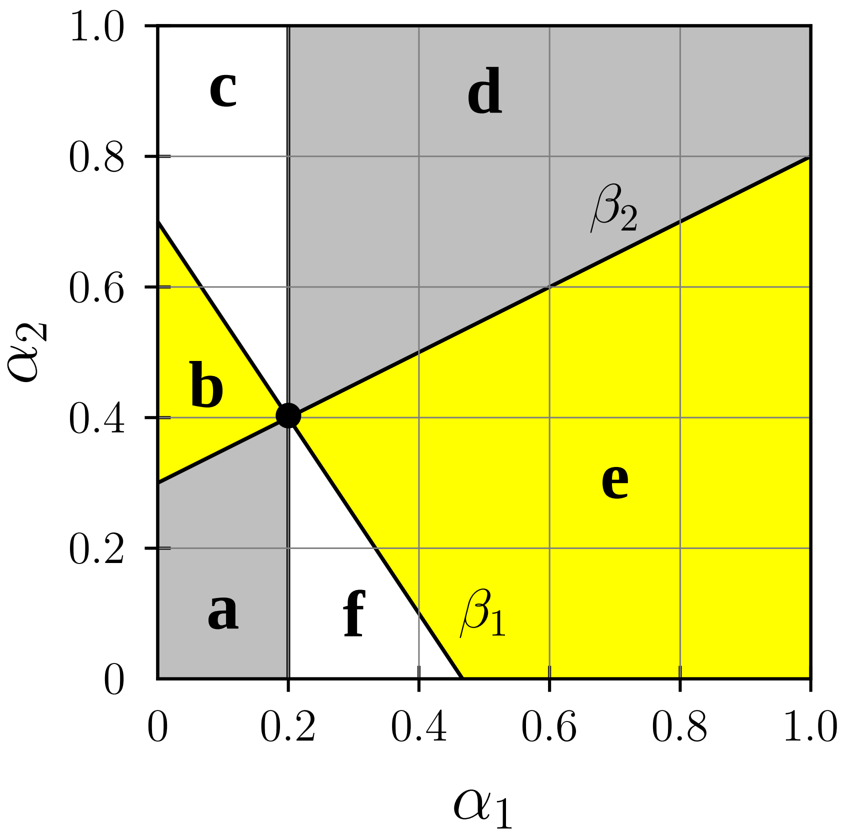
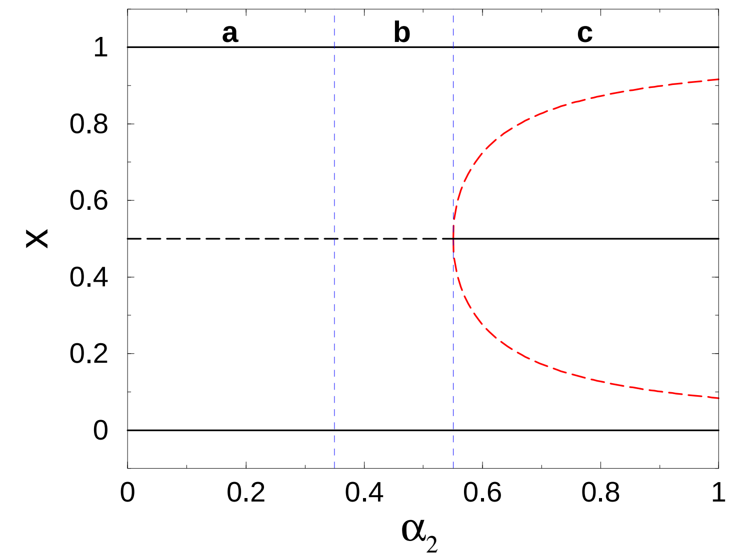
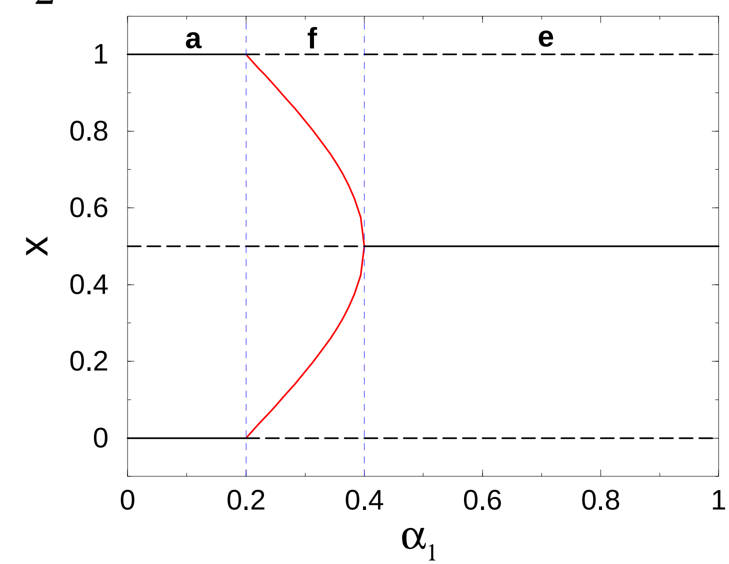
- stability analysis ...



$\alpha_1 = 0.1$



$\alpha_2 = 0.1$



yellow: $x^{(4,5)}$ imaginary
gray: $x^{(4,5)}$ outside (0,1)
(c): $x^{(4,5)}$ unstable

Estimation of Spatial Effects

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- *1. pair approximation:* pairs of nearest neighbor cells σ, σ'
  doublet frequency: $\left\langle x_{\sigma,\sigma'} \right\rangle$
  spatial correlation: $c_{\sigma|\sigma'} := \left\langle x_{\sigma,\sigma'} \right\rangle / \left\langle x_{\sigma'} \right\rangle$

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  doublet frequency: $\langle x_{\sigma,\sigma'} \rangle$
  spatial correlation: $c_{\sigma|\sigma'} := \langle x_{\sigma,\sigma'} \rangle \, / \, \langle x_{\sigma'} \rangle$
  $\Rightarrow$ closed macroscopic dynamics

$$\frac{d}{dt}\langle x(t)\rangle = \sum_{\underline{\sigma}'} \Big[\quad w(1|0,\underline{\sigma}')\,(1-\langle x\rangle) \prod_{j=1}^{m} c_{\sigma_j|\sigma} - w(0|1,\underline{\sigma}')\,\langle x\rangle \prod_{j=1}^{m} c_{\sigma_j|(1-\sigma)} \Big]$$

$$\frac{dc_{1|1}}{dt} = -\frac{c_{1|1}}{\langle x\rangle}\frac{d}{dt}\langle x\rangle + \frac{1}{\langle x\rangle}\frac{d}{dt}\langle x_{1,1}\rangle$$

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2. local neighborhood approximation: 2$^{\text{nd}}$ nearest neighbors

- decompose neighborhood $n = 13$ of cell i
  into 5 overlapping blocks of size 5
  centered around i or its 4 nearest neighbors $i_1, ..., i_4$
  $\Rightarrow$ reduction of the stochastic dynamics

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  (1) various dependencies: majority v., minority v., ...
  (2) *local effects*: influence of neighborhood ...
- stable *coexistence* of opinions possible
  but mostly for *negative* dependence !
- *micro-macro link*: microscopic stochastic description (CA)
  $\Rightarrow$ derivation of macroscopic dynamics
  different approximation levels allow to predict $x(t)$, $c_{1|1}(t)$