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A Binary Homophily Model for Opinion Dynamics

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Abstract— **In this paper we propose a discrete time binary model, based on the homophily social mechanism, that dynamically reduces the cognitive dissonance among the agents in a social network. We show that the binary homophily model can drive an initially structurally unbalanced network towards a socially balanced one. In order to characterise non-structurally balanced equilibrium points, we introduce a (V, Σ)-factorization that finds an interesting interpretation in terms of structurally balanced classes, and can be used to investigate the case of 3 classes and to provide a complete analysis of the convergence to the equilibrium for small-size networks.**

I. Introduction

Social networks are examples of networked systems where the combinations of local interactions generate complex social behaviours. A global property of interest when dealing with social networks is structural balance [5]. Structurally balanced configurations [8] play a major role in the study of opinion dynamics because they correspond to stable relationships in real life social networks (indeed, structural balance is also referred to as social balance [12]), both when dealing with small-size [7] and large-size [5] groups of agents. The study of opinion dynamics, and in particular the understanding of what conditions lead an initially unbalanced configuration towards a stable and balanced one, has been attracting the interest of experts from various fields, ranging from mathematics [6] and economics [9] to sociology [14], [15], for a long time. However, a realistic model aimed at emulating the evolution of opinion dynamics in social networks and its convergence to a bounded, globally stable, socially meaningful equilibrium point is still an attractive research topic. Such models are of significant importance since they allow to make predictions on the opinions evolution and to design control algorithms that drive the system towards social balance.

Moved by these needs and by the vast amount of recent literature available, also related to the control theory field [12], [4], [11], we propose a novel discrete time model whose state variables update according to the homophily mechanism, see [12] and the seminal work by Lazard and Merton [10]. This paper has been inspired by the work of Mei et al. [12], in which the homophily mechanism is presented as a more realistic explanation for the convergence of opinion dynamics to social balance with respect to the influence mechanism, and by the work of Cisneros-Velarde and Bullo [4], in which the concept of influence dynamics is introduced and a game theoretic approach to the update of the mutual opinions of a pair of agents is adopted. We have also taken inspiration from the pioneering work of Heider [8] and the one of Cartwright and Harary [3], in which the concept of social balance has been mathematically formalized in terms of (signed) graph theory for the first time.

In this paper we introduce a binary model for the study of opinion dynamics in social networks and its convergence to structural balance. In the proposed model the state variables, namely the entries of the social network adjacency matrix, represent the mutual opinions of two agents. They are binary variables that take either a positive or a negative value, depending on whether the agents have a positive or a negative opinion of each other, and they update based on the number of (remaining) agents on which the two agents agree/disagree. This amounts to minimising the number of unbalanced triads every pair of agents belongs to, and consequently the cognitive dissonance in the social network. Compared with [12], we have chosen to restrict the values of the mutual opinions of the agents to the binary set $\{-1, 1\}$ in order to study the social dynamics of networks in which the dynamical evolution only depends on whether the relationship between two agents is friendly or antagonistic, while its "intensity" is not relevant for the evolution of the social network, as it happens in politics, sports, games ...

The idea of adjusting mutual relationships between agents based on the number of unbalanced triads has been widely exploited in the past [1], [2], [4]. In the works of Antel et al. [1], [2] the concepts of local triad dynamics (LTD) and constrained triad dynamics (CTD) have been exploited in order to reflect the human propensity to minimize the unbalanced triads in the network they are involved in and to define the update rule for the mutual relationships between agents. In the work of Facchetti et al. [5] an algorithm for the computation of the global level of balance of social networks in a very large scale setting is studied. It is shown that a high degree of skewness of the sign distributions on the nodes of the graph translates in a just apparent disorder, which in fact leads to a high degree of balance.

In this paper we provide necessary and sufficient conditions for a configuration to be an equilibrium point of the binary homophily model. We show that the state-update law can make a structurally unbalanced network converge to a structurally balanced one. We introduce a (V, Σ)-factorization for the special class of symmetric matrices with entries in $\{-1, 1\}$ and unitary diagonal entries, and we make use of it to provide a characterization of non-structurally balanced equilibrium points. This allows to correlate an equilibrium configuration with the structurally balanced maximal classes in the associated graph.

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The paper is organized as follows: in section II a mathematical formalization of the binary homophily model is provided. In section III a characterization of the equilibrium points is proposed with a special focus on the structurally balanced ones. In section IV the (V, Σ)-factorization is introduced, and the convergence to equilibrium for networks of $N = 4$ agents is completely characterised. In section V a characterization of the non-structurally balanced equilibrium points, based on the (V, Σ)-factorization, is given. Section VI concludes the paper.

Notation. Given $k, n \in \mathbb{Z}$, with $k < n$, the symbol $[k, n]$ denotes the integer set $\{k, k+1, \ldots, n\}$. In the sequel, the (i, j)-th entry of a matrix $\mathbf{M}$ is denoted either by m_{ij} or by $[\mathbf{M}]_{ij}$, while the i-th entry of a vector $\mathbf{v}$ by $[\mathbf{v}]_i$.

Given a square matrix $\mathbf{M}$, the notation $\mathrm{diag}(\mathbf{M})$ denotes the diagonal matrix whose diagonal entries are the diagonal entries of $\mathbf{M}$. The notation $\mathbf{M} = \mathbf{M}_1 \oplus \mathbf{M}_2 \cdots \oplus \mathbf{M}_n$ indicates a block diagonal matrix whose diagonal blocks are $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n$.

We let $\mathbf{e}_i$ denote the i-th vector of the canonical basis of $\mathbb{R}^n$, where the dimension n will be clear from the context. By *signed canonical vectors* we will mean all canonical vectors and their opposite, i.e. the set $\{\pm \mathbf{e}_i, i \in [1, n]\}$. The function $\mathrm{sign}(\cdot): \mathbb{R}^{n\times m} \to \{-1, 0, 1\}^{n\times m}$ is the function that maps a real matrix into a matrix taking values in $\{-1, 0, 1\}$ in accordance with the sign of its entries.

An *undirected, signed and unweighted graph* is a triple [13] $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \ldots, N\} = [1, N]$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of arcs (edges), and $\mathcal{A} \in \{-1, 0, 1\}^{N\times N}$ is the *adjacency matrix* of the weighted graph $\mathcal{G}$. An arc (j, i) belongs to $\mathcal{E}$ if and only if $\mathcal{A}_{ij} \neq 0$ and when so it may have either weight 1 or weight -1. As the graph is undirected, (i, j) belongs to $\mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$, and they have the same weight (equivalently $\mathcal{A}$ is a symmetric matrix). Since the adjacency matrix uniquely identifies the graph, in the following $\mathcal{G}(\mathcal{A})$ will denote the graph having $\mathcal{A}$ as adjacency matrix.

The graph $\mathcal{G}$ is *complete* if, for every pair of nodes (i, j), $i, j \in \mathcal{V}$, there is an edge connecting them, namely $(i, j) \in \mathcal{E}$. If so, $\mathcal{E} = \mathcal{V}\times\mathcal{V}$ and $\mathcal{A} \in \{-1, 1\}^{N\times N}$. This will be our steady assumption in the paper. Two vertices i and j are *friends* (*enemies*) if the edge connecting them has positive (negative) weight. Given 3 distinct vertices i, j and $k \in \mathcal{V}$, the *triad* (i, j, k) is called *balanced* [4] if $[\mathcal{A}]_{ij}[\mathcal{A}]_{jk}[\mathcal{A}]_{ki} = 1$ and *unbalanced* if $[\mathcal{A}]_{ij}[\mathcal{A}]_{jk}[\mathcal{A}]_{ki} = -1$.

A graph $\mathcal{G}$ is said *structurally balanced* if it polarizes in two factions of nodes such that nodes in the same faction are friends and nodes from different factions are enemies. The following result easily follows from Lemma 2.2 in [12].

Proposition 1. *Given a matrix $\mathbf{X} \in \{-1, 1\}^{N\times N}$, with unitary diagonal entries, the following facts are equivalent:*

i) $\mathbf{X} = \mathbf{x}\mathbf{x}^\top$, *for some vector* $\mathbf{x} \in \{-1, 1\}^N$*;*
ii) *rank*$(\mathbf{X}) = 1$*;*
iii) *the graph* $\mathcal{G}(\mathbf{X})$ *is structurally balanced;*
iv) *all triads of distinct vertices in* $\mathcal{G}(\mathbf{X})$ *are balanced.*

In the following we will say that $\mathbf{X}$ is structurally balanced if $\mathcal{G}(\mathbf{X})$ is structurally balanced.

II. The binary homophily model

Given a group of $N \geq 3$ agents, we let $x_{ij}(t)$ denote the opinion that agent i has of agent j at the time instant t. We assume that for every $i, j \in [1, N]$, $x_{ij}(t) \in \{-1, 1\}$, where

$$x_{ij}(t) = \begin{cases} 1, & \text{if } i \text{ has a good opinion of } j; \\ -1, & \text{if } i \text{ has a bad opinion of } j. \end{cases}$$

We also assume that $x_{ji}(t) = x_{ij}(t)$. For every pair $(i, j), i \neq j$, we introduce the sets

$$\begin{aligned} \mathcal{A}_{ij}(t) &:= \{k \in [1, N], k \neq i, j : x_{ik}(t)x_{jk}(t) = 1\}, \\ \mathcal{D}_{ij}(t) &:= \{k \in [1, N], k \neq i, j : x_{ik}(t)x_{jk}(t) = -1\}, \end{aligned}$$

representing the sets of agents, distinct from i and j, on which i and j agree or disagree, respectively, at time t. We assume that the relations between pairs of agents are updated according to the following *binary homophily model*, that is:

$$x_{ij}(t+1) = \begin{cases} 1, & \text{if } |\mathcal{A}_{ij}(t)| > |\mathcal{D}_{ij}(t)|; \\ -1, & \text{if } |\mathcal{A}_{ij}(t)| < |\mathcal{D}_{ij}(t)|; \\ x_{ij}(t), & \text{otherwise.} \end{cases} \quad (1)$$

This amounts to assuming that i and j at time $t+1$ will have a good opinion of each other if at time t they agree in their evaluations of most of the other agents. They will have a bad opinion of each other if, on the contrary, they disagree on most of the other elements of the group. If, finally, their opinions coincide on exactly half of the other agents (something that is possible only if the overall number of agents, N, is even) they will keep their mutual evaluations unchanged. The binary homophily model can be equivalently described as:

$$x_{ij}(t+1) = \begin{cases} \mathrm{sign}\Big(\sum_{k\neq i,j} x_{ik}(t)x_{jk}(t)\Big), & \text{if } \sum_{k\neq i,j} x_{ik}(t)x_{jk}(t) \neq 0; \\ x_{ij}(t), & \text{if } \sum_{k\neq i,j} x_{ik}(t)x_{jk}(t) = 0. \end{cases} \quad (2)$$

So, if $\mathbf{X}(t)$ denotes the $N\times N$ symmetric matrix with entries in {-1,1} whose (i, j)-th entry is $x_{ij}(t)$, the binary homophily model can be expressed in matrix form as

$$\begin{aligned} &\mathbf{X}(t+1) \\ &= \mathrm{sign}\Big[\Big(\mathbf{X}(t) - \mathrm{diag}(\mathbf{X}(t))\Big)\Big(\mathbf{X}(t) - \mathrm{diag}(\mathbf{X}(t))\Big)^\top + \alpha\mathbf{X}(t)\Big] \end{aligned} \quad (3)$$

with α arbitrary in $(0, 1)$. The entries $\alpha x_{ij}(t)$ of the term $\alpha\mathbf{X}(t)$ are irrelevant when $[(\mathbf{X}(t) - \mathrm{diag}(\mathbf{X}(t)))(\mathbf{X}(t) - \mathrm{diag}(\mathbf{X}(t)))^\top]_{ij}$ is nonzero, while for $[(\mathbf{X}(t) - \mathrm{diag}(\mathbf{X}(t)))(\mathbf{X}(t) - \mathrm{diag}(\mathbf{X}(t)))^\top]_{ij} = 0$ they ensure that $x_{ij}(t+1) = x_{ij}(t)$.

Remark 2. *It is worth noticing that, after the first iteration of the binary homophily model, the matrix $\mathbf{X}(t)$ is not only symmetric but also with unitary diagonal elements, i.e. $\mathbf{X}(1) = \mathbf{X}(1)^\top$ and $x_{ii}(1) = 1$, $\forall i \in [1, N]$. So, if we define*

$$S_1^N := \{\mathbf{M} = \mathbf{M}^\top \in \{-1, 1\}^{N\times N} \colon [\mathbf{M}]_{ii} = 1, \forall i \in [1, N]\},$$

then $\forall\ \mathbf{X}(0) = \mathbf{X}(0)^\top \in \{-1,1\}^{N\times N}$, $\mathbf{X}(t) \in S_1^N$, $\forall t \geq 1$.

If we assume $\mathbf{X}(0) \in S_1^N$ then, $\forall t \geq 0$, $\mathrm{diag}(\mathbf{X}(t)) = I_N$ and $[\mathbf{X}(t) - \mathrm{diag}(\mathbf{X}(t))]^\top = \mathbf{X}(t) - I_N$. Therefore, under this assumption, the binary homophily model can be equivalently rewritten as

$$\begin{aligned}\mathbf{X}(t+1) &= \mathrm{sign}\Big((\mathbf{X}(t) - I_N)^2 + \alpha\mathbf{X}(t)\Big)\\ &= \mathrm{sign}\Big((\mathbf{X}(t))^2 + \beta\mathbf{X}(t) + I_N\Big),\end{aligned}$$

where $\beta := -2 + \alpha$ is any real number in $(-2,-1)$. Moreover, by noticing that $[\mathbf{X}(t)^2 + \beta\mathbf{X}(t) + I_N]_{ii} = N + \beta + 1$, $\forall t \geq 0$, and $N \geq 3$, it directly follows that

$$\Big[\mathrm{sign}\Big((\mathbf{X}(t))^2 + \beta\mathbf{X}(t) + I_N\Big)\Big]_{ii} = \Big[\mathrm{sign}\Big((\mathbf{X}(t))^2 + \beta\mathbf{X}(t)\Big)\Big]_{ii}.$$

Since the identity matrix does not play any role in the calculation of the off-diagonal entries of the matrix $\mathbf{X}(t)$, the binary homophily model (under the assumption that $\mathbf{X}(0) \in S_1^N$) can be rewritten as:

$$\mathbf{X}(t+1) = \mathrm{sign}\Big((\mathbf{X}(t))^2 + \beta\mathbf{X}(t)\Big), \tag{4}$$

$-2 < \beta < -1$.
In the rest of the paper we will steadily assume $\mathbf{X}(0) \in S_1^N$ and hence we will make use, equivalently, of the update equations (2) and (4).

III. Equilibrium Points Characterization and Structurally Balanced Equilibrium Points

A matrix $\mathbf{X}^* \in S_1^N$ is an *equilibrium point* for the binary homophily model if

$$\mathbf{X}(0) = \mathbf{X}^* \Rightarrow \mathbf{X}(t) = \mathbf{X}^*,\ \forall t \geq 0.$$

From (2) we deduce that $\mathbf{X}^* \in S_1^N$ is an equilibrium point if and only if the following condition holds.

Proposition 3. *A matrix $\mathbf{X}^* \in S_1^N$ is an equilibrium point if and only if $\mathbf{X}^* = \mathrm{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$.*

Proof: We preliminarily notice that, as a result of (2) and of the previous remark, $\mathbf{X}^*$ is an equilibrium point for the binary homophily model if and only if $\forall i,j \in [1,N]$, one has:

$$\begin{aligned} i \neq j \text{ and } \sum_{k\neq i,j} x^*_{ik}x^*_{jk} \neq 0 \quad &\Rightarrow \quad x^*_{ij} = \mathrm{sign}\Big(\sum_{k\neq i,j} x^*_{ik}x^*_{jk}\Big),\\ i = j \quad &\Rightarrow \quad x^*_{ii} = 1. \end{aligned} \tag{5}$$

On the other hand, for every $\mathbf{X}^* \in S_1^N$ condition $\mathbf{X}^* = \mathrm{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$ can be equivalently expressed, for every $i,j, i\neq j$, as

$$\begin{aligned}[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ij} &= \sum_k x^*_{ik}x^*_{jk} - x^*_{ij}\\ &= x^*_{ii}x^*_{ij} + x^*_{ij}x^*_{jj} + \sum_{k\neq i,j} x^*_{ik}x^*_{jk} - x^*_{ij} = x^*_{ij} + \sum_{k\neq i,j} x^*_{ik}x^*_{jk}.\end{aligned}$$

By making use of these preliminary remarks we can prove the result.

Necessity: Let us assume that $\mathbf{X}^*$ is an equilibrium point for the binary homophily model. Then, by making use of the characterization (5), we get $[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ii} = N-1$, therefore $\mathrm{sign}([(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ii}) = 1 = x^*_{ii}$. For the off-diagonal entries we distinguish the following two cases:

$$a)\ \sum_{k\neq i,j} x^*_{ik}x^*_{jk} \neq 0 \qquad b)\ \sum_{k\neq i,j} x^*_{ik}x^*_{jk} = 0.$$

a) If $\sum_{k\neq i,j} x^*_{ik}x^*_{jk} > 0$ then $x^*_{ij} = 1$. On the other hand,

$$[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ij} = x^*_{ij} + \sum_{k\neq i,j} x^*_{ik}x^*_{jk} > 0$$

from which it follows that $[\mathrm{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)]_{ij} = 1 = x^*_{ij}$. Analogous calculations can be done to verify the case $\sum_{k\neq i,j} x^*_{ik}x^*_{jk} < 0$.
b) If $\sum_{k\neq i,j} x^*_{ik}x^*_{jk} = 0$ then

$$[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ij} = x^*_{ij} + \sum_{k\neq i,j} x^*_{ik}x^*_{jk} = x^*_{ij}.$$

Thus, $[\mathrm{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)]_{ij} = x^*_{ij}$.
Therefore, if $\mathbf{X}^*$ is an equilibrium point, then $\mathbf{X}^* = \mathrm{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$.

Sufficiency: Let us suppose that the matrix $\mathbf{X}^* \in S_1^N$ satisfies $\mathbf{X}^* = \mathrm{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$. We will show that $\mathbf{X}^*$ is an equilibrium point for the binary homophily model, namely its entries satisfy the characterization (5). Consider the identity

$$[(\mathbf{X}^*)^2 - \mathbf{X}^*]_{ij} = x^*_{ij} + \sum_{k\neq i,j} x^*_{ik}x^*_{jk}.$$

We distinguish the following cases:
- If $\sum_{k\neq i,j} x^*_{ik}x^*_{jk} \leq -2$, since

$$x^*_{ij} = \mathrm{sign}\Big(x^*_{ij} + \sum_{k\neq i,j} x^*_{ik}x^*_{jk}\Big) \tag{6}$$

we get $x^*_{ij} = -1$.
- If $\sum_{k\neq i,j} x^*_{ik}x^*_{jk} \geq 2$ from (6), we get $x^*_{ij} = 1$.
- If $\sum_{k\neq i,j} x^*_{ik}x^*_{jk} = -1$, it could not happen that $x^*_{ij} = 1$, otherwise one would get $x^*_{ij} + \sum_{k\neq i,j} x^*_{ik}x^*_{jk} = 0$, and so (6) could not hold, against the hypothesis. Therefore it must be $x^*_{ij} = -1$. An analogous reasoning applies to $\sum_{k\neq i,j} x^*_{ik}x^*_{jk} = 1$, in which case $x^*_{ij} = 1$ is obtained.
So, every time $\sum_{k\neq i,j} x^*_{ik}x^*_{jk} \neq 0$ we have $x^*_{ij} = \mathrm{sign}\Big(\sum_{k\neq i,j} x^*_{ik}x^*_{jk}\Big)$, and this proves that $\mathbf{X}^*$ is an equilibrium point. $\square$

Remark 4. *If $\mathbf{X}^* \in S_1^N$ is such that $\mathcal{G}(\mathbf{X}^*)$ is structurally balanced, i.e. (see Proposition 1) there exists $\mathbf{x} \in \{-1,1\}^N$ such that $\mathbf{X}^* = \mathbf{x}\mathbf{x}^\top$, then the condition expressed in Proposition 3 is trivially satisfied.*

For certain values of N, we can show that networks of N agents admit only structurally balanced equilibrium points.

Proposition 5. *Given a network of N agents, with $N \in \{3,4,5,7\}$, if $\mathbf{X}^*$ is an equilibrium point for the binary homophily model, then $\mathcal{G}(\mathbf{X}^*)$ is structurally balanced.*

Proof: We will prove this statement for $N = 3, 5, 7$, i.e., $N = 2p+1$ and $p \in [1, 3]$, by contrapositive. The proof of the case $N = 4$ is similar and hence omitted. If $\mathcal{G}(\mathbf{X}^*)$ is not structurally balanced then (see Proposition 1) there exists at least an unbalanced triad in $\mathcal{G}(\mathbf{X}^*)$. Without loss of generality we will assume that $(1, 2, 3)$ is such a triad. Two cases may occur as Fig. 1 illustrates:

Fig. 1. Unbalanced triads. On the left: a triad with a single negative arc (case a)). On the right: a triad with three negative arcs (case b)).

a) There is only one negative edge in $(1, 2, 3)$. Without loss of generality we will suppose that $x^*_{12} = -1, x^*_{13} = 1, x^*_{23} = 1$. Since $x^*_{12} = -1$, it must hold that $\sum_{k \neq 1,2} x^*_{1k} x^*_{2k} \leq 0$. Moreover, as $x^*_{13} x^*_{23} = 1$, it must hold that $\sum_{k \geq 4} x^*_{1k} x^*_{2k} \leq -1$. Let us define $\mathcal{D}_{12}$, the set of agents on which agents 1 and 2 disagree, namely:

$$\mathcal{D}_{12} := \{k \neq 1, 2 : x^*_{1k} x^*_{2k} = -1\}.$$

It holds that $\mathcal{D}_{12} \subseteq [4, N] = [4, 2p+1]$ and $|\mathcal{D}_{12}| \geq \lceil \frac{2p-1}{2} \rceil = p$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. On the other hand, since $x^*_{13} = 1, x^*_{23} = 1$ and $x^*_{12} = -1$ (and so $x^*_{23} x^*_{12} = -1$), then the set of agents on which agents 1 and 3 agree, namely

$$\mathcal{A}_{13} := \{k \neq 1, 3 : x^*_{1k} x^*_{3k} = 1\},$$

is such that $\mathcal{A}_{13} \subseteq [4, 2p+1]$ and $|\mathcal{A}_{13}| \geq p$. Keeping in mind that $\mathcal{D}_{12} \subseteq [4, 2p+1], \mathcal{A}_{13} \subseteq [4, 2p+1]$, $|\mathcal{D}_{12}| \geq p, |\mathcal{A}_{13}| \geq p$ and that $|[4, 2p+1]| = 2p-2$, it holds that $|\mathcal{D}_{12} \cap \mathcal{A}_{13}| \geq 2$. Moreover, $\forall k \in \mathcal{D}_{12} \cap \mathcal{A}_{13}$ we have that $x^*_{1k} x^*_{2k} = -1, x^*_{1k} x^*_{3k} = 1$, from which it follows that $\forall k \in \mathcal{D}_{12} \cap \mathcal{A}_{13}$, $x^*_{2k} x^*_{3k} = -1$.
Finally, since $x^*_{23} = 1$, the set of the agents on which agents 2 and 3 share the same opinion, $\mathcal{A}_{23}$, is such that $\mathcal{A}_{23} \subseteq [4, N] \setminus (\mathcal{D}_{12} \cap \mathcal{A}_{13})$ and $|\mathcal{A}_{23}| \geq p$, but $|[4, 2p+1] \setminus (\mathcal{D}_{12} \cap \mathcal{A}_{13})| \leq (2p-2) - 2 = 2p-4$ and for $p \in [1, 3]$, the condition $p \leq |\mathcal{A}_{23}| \leq 2p-4$ is impossible. This contradicts the fact that $x^*_{23} = 1$ satisfies the equilibrium condition.

b) All the edges in the triad $(1, 2, 3)$ are negative. Following a reasoning analogous to the one in a), we can show that $\mathcal{D}_{12} \subseteq [4, 2p+1]$ and $|\mathcal{D}_{12}| \geq p$. Similarly, the set $\mathcal{D}_{13}$ of the agents about which agents 1 and 3 disagree must be such that $\mathcal{D}_{13} \subseteq [4, 2p+1]$ and $|\mathcal{D}_{13}| \geq p$. But this leads to conclude that the set $\mathcal{D}_{23}$ of the agents on which agents 2 and 3 disagree satisfies the condition $p \leq |\mathcal{D}_{23}| \leq 2p-4$ that cannot be true for $p \in [1, 3]$, thus contradicting the fact that $\mathbf{X}^*$ is an equilibrium point. $\square$

IV. (V, Σ)-FACTORIZATION

In this section we propose an equivalent representation for the matrices in S_1^N that allows us to derive additional conditions for the analysis of the equilibrium points of the binary homophily model.

Lemma 6. *Given $\mathbf{X} \in S_1^N$, there exist a permutation matrix $\mathbf{P} \in \{0, 1\}^{N \times N}$, positive integers k and n_i, vectors $\mathbf{v}_i \in \{-1, 1\}^{n_i}, i \in [1, k]$, with $\sum_{i=1}^k n_i = N$, and $\mathbf{\Sigma} \in S_1^k$, such that*

$$\mathbf{P}^\top \mathbf{X} \mathbf{P} = \mathbf{V} \mathbf{\Sigma} \mathbf{V}^\top, \text{ where } \mathbf{V} := \mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_k. \quad (7)$$

Proof: We preliminarily notice that, given $\mathbf{X} \in S_1^N$, it is always possible to select distinct rows of $\mathbf{X}$ such that all the other rows are either identical to or the opposite of one of them. This means that $\forall\ \mathbf{X} \in S_1^N, \exists k \leq N$, $\mathbf{B} \in \{-1, 1\}^{k \times N}$, whose rows are pairwise linearly independent, and a matrix $\mathbf{A} \in \{-1, 0, 1\}^{N \times k}$, of rank k, whose rows are signed canonical vectors, such that $\mathbf{X} = \mathbf{A}\mathbf{B}$.

Without loss of generality we can choose a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}^\top \mathbf{X} \mathbf{P} = [\mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_k] \left[\begin{array}{c} \mathbf{b}_1^\top \\ \hline \dots \\ \hline \mathbf{b}_k^\top \end{array}\right], \quad (8)$$

where $\mathbf{v}_i \in \{-1, 1\}^{n_i}, i \in [1, k]$, $n_i \geq 1$, $\sum_{i=1}^k n_i = N$, and $\mathbf{b}_i \in \{-1, 1\}^N, i \in [1, k]$. Partitioning the vectors $\mathbf{b}_i$ in blocks, according to the block partitioning of the matrix on the left in the above factorisation, we get

$$\left[\begin{array}{c} \mathbf{b}_1^\top \\ \hline \dots \\ \hline \mathbf{b}_k^\top \end{array}\right] = \left[\begin{array}{c|c|c|c} \mathbf{v}_{11}^\top & \mathbf{v}_{12}^\top & \dots & \mathbf{v}_{1k}^\top \\ \hline \dots & \dots & \dots & \dots \\ \hline \mathbf{v}_{k1}^\top & \mathbf{v}_{k2}^\top & \dots & \mathbf{v}_{kk}^\top \end{array}\right], \quad (9)$$

with $\mathbf{v}_{ij} \in \{-1, 1\}^{n_j}$. We notice that, since $\mathbf{X} \in S_1^N$, $\mathbf{P}^\top \mathbf{X} \mathbf{P} \in S_1^N$ as well. Therefore the diagonal blocks of dimension $n_i, i \in [1, k]$, of the matrix $\mathbf{P}^\top \mathbf{X} \mathbf{P}$ belong to $S_1^{n_i}$, and this implies $\mathbf{v}_i = \mathbf{v}_{ii}$ for every $i \in [1, k]$. Putting together (8) and (9) and making use of the symmetry of $\mathbf{P}^\top \mathbf{X} \mathbf{P}$, we also obtain $\mathbf{v}_i \mathbf{v}_{ij}^\top = (\mathbf{v}_j \mathbf{v}_{ji}^\top)^\top, \forall i, j \in [1, k], i \neq j$, and due to the fact that the vector components are either 1 or -1, it must be that either (a) $\mathbf{v}_{ji} = \mathbf{v}_i$ and $\mathbf{v}_{ij} = \mathbf{v}_j$ or (b) $\mathbf{v}_{ji} = -\mathbf{v}_i$ and $\mathbf{v}_{ij} = -\mathbf{v}_j$. In light of these considerations we assume $\mathbf{v}_{ij} = \sigma_{ij} \mathbf{v}_j$, with $\sigma_{ij} \in \{-1, 1\}$ and it must be $\sigma_{ij} = \sigma_{ji}$. Therefore we have $\mathbf{P}^\top \mathbf{X} \mathbf{P} = \mathbf{V} \mathbf{\Sigma} \mathbf{V}^T$, where $\mathbf{V} := \mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_k \in \{-1, 0, 1\}^{N \times k}$ and

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & \sigma_{12} & \dots & \sigma_{1k} \\ \sigma_{21} & 1 & \dots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \vdots & 1 \end{bmatrix} \in S_1^k.$$

$\square$

In the following, we will refer to the factorization (7) as a (V, Σ)-*factorization*. As shown in Lemma 6, every matrix $\mathbf{X} \in S_1^N$ admits a (V, Σ)-factorization, modulo a suitable permutation of its rows and columns, and we will provide characterisations of the matrices $\mathbf{X}^* \in S_1^N$ that represent equilibrium points of the binary homophily model in terms of the matrix $\mathbf{\Sigma}$ and of the sizes n_i of the vectors $\mathbf{v}_i$ appearing in $\mathbf{V}$ involved in any such factorisation. On the contrary, the specific entries of the vectors $\mathbf{v}_i$ will play no role.
Finally, note that the permutation matrix $\mathbf{P}$ such that $\mathbf{P}^\top \mathbf{X}^* \mathbf{P} = \mathbf{V} \mathbf{\Sigma} \mathbf{V}^\top$ is not relevant when providing such

a characterisation, since $\mathbf{X}^*$ is an equilibrium point if and only if $\mathbf{P}^\top \mathbf{X}^* \mathbf{P}$ is an equilibrium point. Therefore in the following we will assume, for the sake of simplicity, $\mathbf{X}^* = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$.

We now propose a graph interpretation of a (V, Σ)-factorization. To this aim it is worth noticing that the identity $\mathbf{X}^* = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$ can be equivalently expressed as

$$\mathbf{X}^* = \begin{bmatrix} \mathbf{v}_1\mathbf{v}_1^\top & \sigma_{12}\mathbf{v}_1\mathbf{v}_2^\top & \dots & \sigma_{1k}\mathbf{v}_1\mathbf{v}_k^\top \\ \sigma_{12}\mathbf{v}_2\mathbf{v}_1^\top & \mathbf{v}_2\mathbf{v}_2^\top & \dots & \sigma_{2k}\mathbf{v}_2\mathbf{v}_k^\top \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1k}\mathbf{v}_k\mathbf{v}_1^\top & \sigma_{2k}\mathbf{v}_k\mathbf{v}_2^\top & \dots & \mathbf{v}_k\mathbf{v}_k^\top \end{bmatrix}. \quad (10)$$

From this expression one deduces that the product $\mathbf{v}_i\mathbf{v}_i^\top$, $i \in [1, k]$, corresponds to a structurally balanced subclass, let us call it $\mathcal{C}_i$, of cardinality $n_i = \dim(\mathbf{v}_i)$, in the graph $\mathcal{G}(\mathbf{X}^*)$. Based on the sign of the entries of $\mathbf{v}_i$, the class $\mathcal{C}_i$ splits into two adverse factions, $\mathcal{C}_{iA}$ and $\mathcal{C}_{iB}$, each of them consisting of agents that are friends. On the other hand, σ_{ij} can be interpreted as the relation between agents in class $\mathcal{C}_i$ and agents in class $\mathcal{C}_j$. Specifically, all agents in a faction $\mathcal{C}_{iA}$ are friends *[enemies]* of all agents of $\mathcal{C}_{jA}$ and enemies *[friends]* of all agents of $\mathcal{C}_{jB}$ provided that σ_{ij} is positive *[negative]*, and the same statement holds true if the suffixes A and B are swapped. As a result, in the partition of $\mathcal{G}(\mathbf{\Sigma})$ thus obtained, $\forall i, j \in [1, k]$, $\mathcal{G}(\mathcal{C}_i \cup \mathcal{C}_j)$ is structurally balanced, in turn. Figure 2 is a graphical representation of what has just been stated above. As the graph $\mathcal{G}(\mathbf{X}^*)$ is complete and unweighted we will draw only the positive edges within the vertices of each class $\mathcal{C}_i$, while negative edges will be omitted. Self-loops will always have weight 1 and will be omitted, in turn. Arcs between two distinct classes $\mathcal{C}_i$ and $\mathcal{C}_j$ will be represented by means of the parameter σ_{ij}, as a result of the previous interpretation.

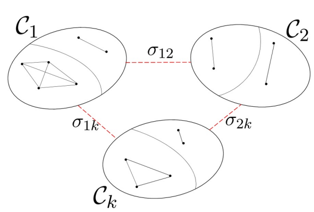

Fig. 2. Graphic representation of $\mathcal{G}(\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top)$.

We now introduce a technical lemma that will be used in the following.

Lemma 7. *Let $\mathbf{v}_i \in \{-1, 1\}^{n_i}$, $i \in [1, k]$, and set $\mathbf{V} = \mathbf{v}_1 \oplus \dots \otimes \mathbf{v}_k$. Then for every $\mathbf{\Phi} \in S_1^k$ and every matrix $\mathbf{\Psi} \in \mathbb{R}^{k\times k}$, $\mathbf{V}\mathbf{\Phi}\mathbf{V}^\top = \mathrm{sign}(\mathbf{V}\mathbf{\Psi}\mathbf{V}^\top) \iff \mathbf{\Phi} = \mathrm{sign}(\mathbf{\Psi})$.*

Proof: For every $i, j \in [1, N]$ we have $\mathbf{e}_i^\top \mathbf{V}\mathbf{\Phi}\mathbf{V}^\top \mathbf{e}_j = \mathrm{sign}(\mathbf{e}_i^\top \mathbf{V}\mathbf{\Psi}\mathbf{V}^\top \mathbf{e}_j)$ if and only if $\forall \ell, s \in [1, k]$, $\forall r \in [1, n_\ell], p \in [1, n_s]$ condition $[\mathbf{v}_\ell]_r \mathbf{e}_\ell^\top \mathbf{\Phi}\mathbf{e}_s [\mathbf{v}_s]_p = \mathrm{sign}([\mathbf{v}_\ell]_r \mathbf{e}_\ell^\top \mathbf{\Psi}\mathbf{e}_s [\mathbf{v}_s]_p)$ holds, which means that $\forall \ell, s \in [1, k]$ we have $\mathbf{e}_\ell^\top \mathbf{\Phi}\mathbf{e}_s = \mathrm{sign}(\mathbf{e}_\ell^\top \mathbf{\Psi}\mathbf{e}_s)$. ☐

The following proposition provides a condition on a matrix $\mathbf{X}_0 \in S_1^N$ that guarantees that the binary homophily model starting from $\mathbf{X}_0$ converges to a structurally balanced equilibrium point in one step. Such a condition relies on the matrix $\mathbf{\Sigma}$ involved in a (V, Σ)-factorization of $\mathbf{X}_0$.

Proposition 8. *Consider a matrix $\mathbf{X}_0 \in S_1^N$ and a (V, Σ)-factorization of $\mathbf{X}_0$, i.e., $\mathbf{X}_0 = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$, where $\mathbf{V} := \mathbf{v}_1 \oplus \mathbf{v}_2 \dots \oplus \mathbf{v}_k$, $\mathbf{v}_i \in \{-1, 1\}^{n_i}$, $i \in [1, k]$, $\sum_{i=1}^k n_i = N$, and $\mathbf{\Sigma} \in S_1^k$. Set*

$$\mathbb{N} := \mathbf{V}^\top \mathbf{V} = n_1 \oplus n_2 \oplus \dots \oplus n_k. \quad (11)$$

If $\mathbf{\Sigma}$ satisfies

$$\mathrm{sign}\left(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \frac{3}{2}\mathbf{\Sigma}\right) = \mathbf{w}\mathbf{w}^\top \quad (12)$$

for some $\mathbf{w} \in \{-1, 1\}^k$, then $\mathrm{sign}\Big(\mathbf{X}_0^2 - \frac{3}{2}\mathbf{X}_0\Big) = \mathbf{V}\mathbf{w}\mathbf{w}^\top\mathbf{V}^\top$. Therefore the binary homophily model starting from $\mathbf{X}(0) = \mathbf{X}_0$ converges in one step to the structurally balanced equilibrium point $\mathbf{X}^* = \mathbf{v}\mathbf{v}^\top$, where $\mathbf{v} := \mathbf{V}\mathbf{w}$. In particular, if all entries of $\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \frac{3}{2}\mathbf{\Sigma}$ are positive then the equilibrium point is $\mathbf{X}^* = \mathbf{v}\mathbf{v}^\top$, where $\mathbf{v} := \mathbf{V}\mathbf{1}_k$.

Proof: By Lemma 7, if identity (12) holds then

$$\begin{aligned} \mathbf{V}\mathbf{w}\mathbf{w}^\top\mathbf{V}^\top &= \mathrm{sign}\Big(\mathbf{V}\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma}\mathbf{V}^\top - \frac{3}{2}\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top\Big) \\ &= \mathrm{sign}\Big(\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top - \frac{3}{2}\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top\Big) \\ &= \mathrm{sign}\Big(\mathbf{X}_0^2 - \frac{3}{2}\mathbf{X}_0\Big). \end{aligned}$$

On the other hand, the binary homophily model (4) for $\beta = -3/2$ leads to saying that $\mathbf{X}(1) = \mathrm{sign}\Big(\mathbf{X}(0)^2 - \frac{3}{2}\mathbf{X}(0)\Big)$. Therefore if $\mathbf{X}(0) = \mathbf{X}_0$ then $\mathbf{X}(1) = (\mathbf{V}\mathbf{w})(\mathbf{V}\mathbf{w})^\top$, and this concludes the proof. ☐

The following proposition states that when dealing with binary homophily models of size $N = 4$, every $\mathbf{X}_0 \in S_1^4$ is either a (structurally balanced) equilibrium point or it converges in one step to a structurally balanced equilibrium point (see Proposition 5).

Proposition 9. *For every $\mathbf{X}_0 \in S_1^4$, the matrix $\mathbf{X}^* := \mathrm{sign}\left(\mathbf{X}_0^2 - \frac{3}{2}\mathbf{X}_0\right)$ is an equilibrium point for the binary homophily model, and hence it is structurally balanced.*

Proof: If $\mathbf{X}_0$ is structurally balanced, then $\mathbf{X}_0$ is already an equilibrium point, and hence if we assume $\mathbf{X}(0) = \mathbf{X}_0$ then, by adopting model (4) with $\beta = -3/2$, we get $\mathbf{X}_0 = \mathbf{X}(1) = \mathrm{sign}\left(\mathbf{X}(0)^2 - \frac{3}{2}\mathbf{X}(0)\right)$.
Suppose, now, that $\mathbf{X}_0$ is not structurally balanced, and hence it admits a (V, Σ)-factorization $\mathbf{X}_0 = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$, with $\mathbf{\Sigma} \in S_1^k$ for some $k \in \{2, 3, 4\}$ not structurally balanced. We first note that all matrices in S_1^2 are structurally balanced, and hence we have to rule out the case $k = 2$ because it would correspond to a structurally balanced $\mathbf{\Sigma}$ and hence to a structurally balanced $\mathbf{X}_0$. For $k = 3$ assume w.l.o.g. that

$$\mathbb{N} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}.$$

It is straightforward to prove that if $\mathbf{\Sigma} \in S_1^3$ is not structurally balanced, then

$$\operatorname{sign}\left(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \frac{3}{2}\mathbf{\Sigma}\right) = \begin{bmatrix} 1 & -\sigma_{12} & \sigma_{13} \\ -\sigma_{12} & 1 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 1 \end{bmatrix}$$

and this is a structurally balanced matrix. Therefore $\operatorname{sign}\left(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \frac{3}{2}\mathbf{\Sigma}\right) = \mathbf{w}\mathbf{w}^\top$ for some $\mathbf{w} \in \{-1,1\}^3$. So, by applying Proposition 8 we obtain the result.
Finally, if $k = 4$ (and hence $\mathbb{N} = I_4$), it can be proved that if $\mathbf{\Sigma}$ is not structurally balanced then there exists $\mathbf{w} \in \{-1,1\}^4$ such that $\mathbf{\Sigma} = 2I_4 - \mathbf{w}\mathbf{w}^\top$. But then

$$\begin{aligned}\operatorname{sign}\left(\mathbf{\Sigma}^2 - \frac{3}{2}\mathbf{\Sigma}\right) &= \operatorname{sign}[(2I_4 - \mathbf{w}\mathbf{w}^\top)^2 - \frac{3}{2}(2I_4 - \mathbf{w}\mathbf{w}^\top)] \\ &= \operatorname{sign}(I_4 + \frac{3}{2}\mathbf{w}\mathbf{w}^\top) = \mathbf{w}\mathbf{w}^\top.\end{aligned}$$

Therefore, by applying again Proposition 8, we obtain the result. ☐

V. Not structurally balanced equilibria

For $N \geq 6$ there exist equilibrium points $\mathbf{X}^* \in S_1^N$ associated to a not structurally balanced graph $\mathcal{G}(\mathbf{X}^*)$. In the following we provide an example for the binary homophily model of dimension $N = 6$.

Example 1. *It is easy to verify that*

$$\mathbf{X}^* = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix} \in S_1^6$$

satisfies the condition given in Proposition 3, and therefore it is an equilibrium point. It is worth noticing that $\operatorname{rank}(\mathbf{X}^*) = 3$ *(and hence $\mathbf{X}^*$ is not structurally balanced, see Proposition 1) and $\mathbf{P}^\top\mathbf{X}^*\mathbf{P} = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$, where $\mathbf{V} := \mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \mathbf{v}_3$, with*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Proposition 10 below states a necessary and sufficient condition for $\mathbf{X}^*$ to be an equilibrium point in terms of any matrix $\mathbf{\Sigma}$ involved in a (V, Σ)-factorization of $\mathbf{X}^*$.

Proposition 10. *A matrix $\mathbf{X}^* \in S_1^N$ is an equilibrium point for the binary homophily model if and only if*

$$\mathbf{\Sigma} = \operatorname{sign}(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \mathbf{\Sigma}), \tag{13}$$

where $\mathbf{\Sigma} \in S_1^k$ is the matrix $\mathbf{\Sigma}$ involved in a (V, Σ)-factorization of $\mathbf{X}^*$, $\mathbb{N}$ is defined as in (11) and $n_1, \dots, n_k$ are the sizes of the vectors $\mathbf{v}_i$ appearing in $\mathbf{V}$.

Proof: By Proposition 3, $\mathbf{X}^*$ is an equilibrium point if and only if $\mathbf{X}^* = \operatorname{sign}((\mathbf{X}^*)^2 - \mathbf{X}^*)$. From the identity $\mathbf{X}^* = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$, the previous equilibrium condition can be equivalently written as $\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top = \operatorname{sign}(\mathbf{V}(\mathbf{\Sigma}\mathbf{V}^\top\mathbf{V}\mathbf{\Sigma} - \mathbf{\Sigma})\mathbf{V}^\top)$. By Lemma 7, this identity is true if and only if (13) holds. Therefore $\mathbf{X}^*$ is an equilibrium point if and only if (13) holds. ☐

As a consequence of Proposition 10, we can show that if $\mathbf{\Sigma} \in S_1^3$ satisfies (13), it is always possible to find an equilibrium network $\mathbf{X}^*$ and a matrix $\mathbf{V}$ such that $\mathbf{X}^* = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$. The proof is omitted due to page constraints.

Proposition 11. *If $\mathbf{\Sigma} \in S_1^3$, then one can always find positive integers n_1, n_2, n_3 such that*

$$\mathbf{\Sigma} = \operatorname{sign}(\mathbf{\Sigma}\mathbb{N}\mathbf{\Sigma} - \mathbf{\Sigma}), \tag{14}$$

where $\mathbb{N} := n_1 \oplus n_2 \oplus n_3$.

VI. Conclusions

In this paper a binary discrete time homphily model for the opinion dynamics in social networks has been proposed. Necessary and sufficient conditions for a configuration to be an equilibrium point have been provided and we have shown that all structurally balanced configurations are equilibrium points. However there can be also structurally unbalanced equilibrium points. Upon introducing the (V, Σ)-factorization for the class of symmetric matrices with entries in $\{-1, 1\}$ and unitary diagonal entries, we have provided a new characterization of all the equilibrium points. This has also allowed us to show that social networks of sufficiently large size N always exhibit equilibrium points whose graph splits into three structurally balanced classes.

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