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A Bandwagon Bias Based Model for Opinion Dynamics: Intertwining between Homophily and Influence Mechanisms

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ABSTRACT

Recently a model for the interplay between homophily-based appraisal dynamics and influence-based opinion dynamics has been proposed. The model explores for the first time how the opinions of a group of agents on a certain number of issues/topics is influenced by the agents' mutual appraisal and, conversely, the agents' mutual appraisal is updated based on the agents' opinions on the various issues, according to a homophily model. In this paper we show that a simplified (and, in some situations, more feasible) version of the model, that accounts only for the signs of the agents' appraisals rather than for their numerical values, provides an equally accurate and effective model of the opinion dynamics in small networks. The equilibria reached by this model correspond, almost surely, to situations in which the agents' network is complete and structurally balanced. On the other hand, we ensure that such equlibria can always be reached in a finite number of steps, and, differently from the original model, we rule out other types of equilibria that correspond to disconnected social networks.

1. Introduction

Over the last few decades, the modelling and analysis of sociological phenomena have attracted the interests of researchers from various fields, such as sociology, economics, and mathematics [3,9,10,15]. In several cases, sociological models represent the primary focus of the investigation, but there are numerous contexts, such as product promotion, spread of diseases, resource allocation, etc., where social dynamics represents the context in which other phenomena evolve. Consequently, understanding its behaviour represents a preliminary but fundamental step in order to investigate and understand the evolution of the process of interest [1,17]. As a result, it becomes of great importance to build a reliable model for the social dynamics, that allows to forecast the network evolution and thus to design strategies aimed at driving the network towards the desired configuration [22]. Dynamic social balance theory is concerned with the study and analysis of the evolution of socially unbalanced networks towards socially balanced ones, namely networks in balanced configurations in which all the agents split in (at most) two groups in such a way that all the agents in the same group have friendly relationships, while agents from different groups have not [12,14].

Even if, from a modeling perspective, the study of social balance has rather remote origins, as witnessed by the pioneering works of Heider [14], Cartwright and Harary [11,12], DeGroot [7], the dynamic social balance theory represents an active and timely research topic. In this regard we mention the recent works of Mei et al. [19] in which two dynamical models based on two different social mechanisms, the homophily mechanism and the influence mechanism, are proposed. In the homophily mechanism, individuals update their mutual appraisals based on their appraisals of the other group members. In the influence mechanism, instead, each agent attributes an influence to the other network members, based on the appraisal that the agent has about them. Reference [19] shows that both mechanisms drive the network towards social balance, but the homophily mechanism gives a more general explanation for the emergence of the social balance with respect to the influence one.

A relevant contribution to the dynamic social balance analysis is the one from Quattrocchi et al. [23] whose model takes into account the presence of media and gossip as separate mechanisms. Another inspiring work in which a sociological mathematical model, including two coexisting social mechanisms, is studied, is the recent work from Liu et al. [16].

In [16] a novel model in which the interpersonal appraisals and the individual opinions evolve according to an intertwined dynamics has been proposed for the first time. In the proposed state space model, the authors assume as state variables both the in-

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terpersonal appraisals and the agent's opinions, namely the opinions of the agents on a specific selection of topics. Specifically, the model relies on the assumption that the opinion that an agent has on a particular issue is the (signed) weighted average of the opinions that all the other agents have on that issue, where the weights are the appraisals that the agent has about each of them. At the same time, the interpersonal relationship of an agent pair depends on the comparison between the opinions that the two agents have about all the topics into play, thus following a homophily mechanism. This model can be interpreted as a mathematical formalization of a form of cognitive bias known in psychology as "bandwagon bias" [21], by this meaning that our opinions on topics and issues are influenced by the opinions that other individuals have on the same topics and by the relationships we have with those individuals. Bandwagon bias results in an intertwined dynamics involving both a homophily mechanisms for the interpersonal relationships and an influence mechanism for the agents' opinions. This is in line with the fact that, in real life, interpersonal appraisals influence individual opinions and viceversa. Another dynamical model that studies how cognitive bias drives the formation of social influences can be found in [4].

Inspired by [16] we propose here a mixed-binary and real-valued version of the aforementioned model, by this meaning that while we assume that the agent's opinions take (positive or negative) real values, that represent their levels of appreciation or dislike of each specific issue, we do not quantify the level of mutual appraisal, but only take into account whether the mutual relationships between pairs of agents are friendly or hostile. This simplified assumption has been adopted in our previous work [8], as well as in the works of Cisneros-Velarde et al. [5] and of Mei et al. [18] in which mutual appraisals are treated as binary variables. In particular, in [18] the evolution of a signed unweighted non-all-to-all network towards the straightforward generalization of the concept of structural balance has been proposed, thus leading to the graph-theoretical concept of "triad-wise structural balance", where each agent's ego-networks satisfies the structural balance property. This discrete-time "gossip-like" model enjoys the property of convergence towards a non-all-to-all structural balance configuration, while the structure of the associated graph is time invariant, since only the signs of the weights change. In [5] a network formation game, in which pairs of rational individuals strategically change the signs of the edges in a complete network is proposed. The game is shown to strategically reduce the cognitive dissonance in the network along time, by driving the network towards clustering balance [6]. The motivation behind the study of the dynamical evolution of unweighted signed social networks, that units the aforementioned works, comes from the fact that there are many circumstances in which recognizing the type (friendly or hostile) of relationship between individuals is easy, while assessing its intensity is complicated and prone to model errors. In fact, while individual evaluations of certain products or their opinions on certain topics can be easily obtained, attributing numerical values to the mutual appraisals is more challenging and oftentimes individuals prefer to not even reveal them.

We show that our simpler model retains all the good properties of the model proposed in [16] both in terms of transient behaviour and convergence to structurally balanced equilibria, meanwhile strengthening some of the results derived for that model. In particular, our model exhibit only two types of long term behavior: either the social network converges in finite time towards a socially balanced all-to-all equilibrium or asymptotically converges to zero. Other equilibrium structures, that arise for the model in [16] and that correspond to the case when the group of agents splits into disconnected structurally balanced subnetworks, are ruled out by our model assumptions, which are designed for small networks. In such contexts, getting an all-to-all equilibrium network is realistic and, as it will be clear from simulations, the situation when a structurally balanced equilibrium cannot be found and all individuals eventually weaken their opinions and appraisal to avoid long term conflicts (see [3]) is a very rare occurrence.

The paper is organized as follows: in Section 2 the model is introduced and its equilibrium conditions are studied, Section 3 deals with the finite time behaviour of the model, while in Section 4 its asymptotic convergence properties are studied. Section 6 concludes the paper.

Notation. Given $k, n \in \mathbb{Z}$, with $k < n$, the symbol $[k, n]$ denotes the integer set $\{k, k+1, \dots, n\}$. We let $\mathbf{e}_i$ denote the i-th vector of the canonical basis of $\mathbb{R}^n$, where the dimension n will be clear from the context. The vectors $\mathbf{1}_n$ and $\mathbf{0}_n$ denote the n-dimensional vectors whose entries are all 1 or 0, respectively.

The function $\mathrm{sgn}: \mathbb{R}^{n\times m} \to \{-1, 0, 1\}^{n\times m}$ is the function that maps a real matrix into a matrix taking values in $\{-1, 0, 1\}$, in accordance with the sign of its entries.

In the sequel, the (i, j)-th entry of a matrix $\mathbf{X}$ is denoted either by X_{ij} or by $[\mathbf{X}]_{ij}$, while the i-th entry of a vector $\mathbf{v}$ either by v_i or by $[\mathbf{v}]_i$. The notation $\mathbf{X} = \mathrm{diag}\{x_1, x_2, \dots, x_N\}$ indicates the diagonal matrix whose diagonal entries are $x_1, x_2, \dots, x_N$. Given a matrix $\mathbf{X} \in \mathbb{R}^{N\times N}$, the *spectrum* of $\mathbf{X}$, $\sigma(\mathbf{X})$, is the set of eigenvalues of $\mathbf{X}$.

An *undirected and signed graph* is a triple [20] $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \dots, N\} = [1, N]$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ the set of arcs (edges), and $\mathcal{A} \in \{-1, 0, 1\}^{N\times N}$ the *adjacency matrix* of the graph $\mathcal{G}$. An arc (j, i) belongs to $\mathcal{E}$ if and only if $\mathcal{A}_{ij} \neq 0$ and when so it may have either weight 1 or weight -1. As the graph is undirected, (i, j) belongs to $\mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$, and they have the same weight (equivalently $\mathcal{A}$ is a symmetric matrix). A sequence $j_1 \leftrightarrow j_2 \leftrightarrow j_3 \leftrightarrow \cdots \leftrightarrow j_k \leftrightarrow j_{k+1}$ is a *path* of length k connecting j_1 and j_{k+1} provided that $(j_1, j_2), (j_2, j_3), \dots, (j_k, j_{k+1}) \in \mathcal{E}$. A closed path in which each node, except the start-end node, is distinct is called *cycle*, and a cycle of unitary length is also known as *self-loop*. Since the adjacency matrix uniquely identifies the graph, in the following we will oftentimes use the notation $\mathcal{G}(\mathcal{A})$ to denote the graph having $\mathcal{A} \in \{-1, 0, 1\}^{N\times N}$ as adjacency matrix.

The graph $\mathcal{G}$ is said to be *complete* if, for every pair of vertices (i, j), $i, j \in \mathcal{V}$, there is an edge connecting them, namely $(i, j) \in \mathcal{E}$. If so, $\mathcal{E} = \mathcal{V}\times\mathcal{V}$ and $\mathcal{A} \in \{-1, 1\}^{N\times N}$. Given three distinct vertices i, j and $k \in \mathcal{V}$, the *triad* (i, j, k) is said to be *balanced* [5] if $\mathcal{A}_{ij}\mathcal{A}_{jk}\mathcal{A}_{ki} = 1$ and *unbalanced* if $\mathcal{A}_{ij}\mathcal{A}_{jk}\mathcal{A}_{ki} = -1$.

In this work we consider undirected and signed graphs with unitary self loops. Therefore the adjacency matrix of the graph belongs to the set [8]

$$\mathcal{S}_1^N := \{\mathbf{M} \in \{-1, 0, 1\}^{N\times N} : \mathbf{M} = \mathbf{M}^\top,\ M_{ii} = 1,\ \forall i \in [1, N]\}. \quad (1)$$

A graph $\mathcal{G}$ is said to be *structurally balanced* if it can be partitioned into two factions of vertices such that edges between vertices of the same faction have nonnegative weights, and edges between vertices from different factions have nonpositive weights (see, also, the aforementioned concept of "triad-wise structural balance" in [18]).

The following result easily follows from Lemma 2.2 in [19].

Lemma 1 (Structural balance for complete graphs)**.** *Given a matrix $\mathbf{X} \in \mathcal{S}_1^N \cap \{-1, 1\}^{N\times N}$, the following facts are equivalent:*

i) $\mathbf{X} = \mathbf{p}\mathbf{p}^\top$, *for some vector* $\mathbf{p} \in \{-1, 1\}^N$*;*
ii) *rank*$(\mathbf{X}) = 1$*;*
ii) *for every* $a, b \in [1, N]$ *either* $\mathbf{e}_a^\top\mathbf{X} = \mathbf{e}_b^\top\mathbf{X}$ *or* $\mathbf{e}_a^\top\mathbf{X} = -\mathbf{e}_b^\top\mathbf{X}$ *;*
iv) *the graph* $\mathcal{G}(\mathbf{X})$ *is structurally balanced;*
v) *all the triads* (i, j, k) *of distinct vertices in* $\mathcal{G}(\mathbf{X})$ *are balanced.*

In the following we will say that $\mathbf{X}$ is structurally balanced if $\mathcal{G}(\mathbf{X})$ is structurally balanced.

2. The model: properties, equilibrium points and periodic solutions

Given a group of N agents, we denote by $\mathbf{X}(t) \in \{-1, 0, 1\}^{N\times N}$ the *appraisal matrix at time t* of the agents, whose (i, j)-th entry represents agent i's appraisal of agent j at time t. $X_{ij}(t) = 1$ if i has positive feelings towards j and $X_{ij}(t) = -1$ if i has negative feelings towards j, while $X_{ij}(t) = 0$ if i chooses not rely on j in forming its opinion[1]. We assume that for each pair of agents (i, j) at each time instant t the appraisal is mutual, namely $X_{ij}(t) = X_{ji}(t)$ $\forall i, j \in [1, N]$, and hence $\mathbf{X}(t)$ is a symmetric matrix $\forall t \geq 0$. The (undirected and signed) graph $\mathcal{G}(\mathbf{X})$, having $\mathbf{X}$ as adjacency matrix, represents the *appraisal network* [19].

We assume that the agents express their opinions about a certain number, say m, of issues. This information is collected in a matrix $\mathbf{Y}(t) \in \mathbb{R}^{N\times m}$, whose (i, j)-th entry is the opinion that agent i has about the issue j at the time instant t. $\mathbf{Y}(t)$ is called the *opinion matrix at the time instant t* of the social network. We assume that the opinion matrix and the appraisal matrix evolve according to an intertwined dynamics expressed by the following equations

$$\mathbf{X}(t+1) = \text{sgn}(\mathbf{Y}(t)\mathbf{Y}(t)^\top) \tag{2}$$

$$\mathbf{Y}(t+1) = \frac{1}{N}\mathbf{X}(t+1)\mathbf{Y}(t) \tag{3}$$

that component-wise correspond to

$$X_{ij}(t+1) = \text{sgn}\left(\sum_{k=1}^{m} Y_{ik}(t)Y_{jk}(t)\right) \tag{4}$$

$$Y_{ij}(t+1) = \frac{1}{N}\sum_{k=1}^{N} X_{ik}(t+1)Y_{kj}(t). \tag{5}$$

Eq. (5) shows that the opinion that agent i has about issue j at the time instant $t+1$ is a (signed) weighted average of the opinions that all agents have about the topic j at the time instant t, where the weights are the appraisals that agent i has about them at the time instant t, divided by the number of agents.

On the other hand, from Eq. (4), we notice that the (i, j)-th entry of the appraisal matrix at the time instant $t+1$, namely, the appraisal that agent i has about agent j at the time instant $t+1$, depends on the comparison between the opinions that agents i and j have about all the topics at the time instant t. In particular, if the agents agree (resp. disagree) on a specific issue k, this will give a positive (resp. negative) contribution $Y_{ik}(t)Y_{jk}(t) > 0$ (resp. $Y_{ik}(t)Y_{jk}(t) < 0$), in determining the relationship between i and j at the time instant $t+1$.

Essentially, this model captures the evolution of opinion-dependent time-varying graph structures. In this regard one can see analogies with the pioneering work form Hagselmann-Krause [13], in which the closeness of opinions determines the structure topology of the (unweighted) interaction graph. On the other hand, in our model all agents potentially communicate and their opinions will rather determine the type (friendly/antagonistic) of relationship. Eqs. (2) and (3) can be grouped into a single equation that describes the update of the opinion matrix alone and takes the form

$$\mathbf{Y}(t+1) = \frac{1}{N}\text{sgn}(\mathbf{Y}(t)\mathbf{Y}(t)^\top)\mathbf{Y}(t). \tag{6}$$

[1] Since we consider small-medium size networks, this formalizes the case when agent i knows agent j but does not find correlation between its own choices and agent j's opinions, and hence chooses not to give it any weight.

Eq. (6) shows that the mathematical abstraction of the bandwagon bias leads the intertwining between opinion dynamics and appraisal dynamics to a peculiar form of opinion dynamics model. It is immediate to notice that if $\mathbf{Y}(0)$ has a zero row (a situation that formalizes the case when one of the agents expresses no opinion on any of the m topics), then that same row remains zero in every subsequent opinion matrix $\mathbf{Y}(t), t \geq 0$. Similarly, if $\mathbf{Y}(0)$ has a zero column (none of the agents expresses any judgement on a specific topic), that same column remains zero in all the matrices $\mathbf{Y}(t), t \geq 0$. Therefore both cases are of no interest (substantially, one can always remove the agent and/or the topic and focus on the analysis of the remaining variables) and will not be considered in the following.

Remark 2. Compared with the model proposed and investigated in [16], we have modified the law that governs the appraisal matrix update and how it affects the opinion dynamics in two aspects. First, we have chosen to keep into account only the signs of the mutual appraisals, rather than their absolute values. This is motivated by the fact that, in a lot of practical situations, being able to assess the sign of the mutual appraisal is easier and more robust to modeling errors with respect to determining the numerical value associated to the tie strength. Moreover, the influence that agent j can have on the opinion agent i has on a certain issue does not necessarily scale with the absolute value of X_{ij}. Secondly, we have chosen to "give a weight" also to the fact that a pair of agents chooses not rely on each other's opinion, namely to the fact that $X_{ij} = 0$. Since we consider small-medium size networks, this formalizes the case when agent i knows agent j but does not find correlation between its own choices and agent j's opinions, and hence chooses not to give it any weight. In this perspective, the fact that the mutual appraisal is 0 is an information that should be considered and this motivates the fact that in the opinion dynamics update equation (5) each row is divided by the overall number of agents N, rather than by the absolute value of its entries. It is worth noticing that, however, since the appraisal matrix is obtained by comparing the (real valued) opinions of the agents on the various topics into play, and its (i, j)-th entry is zero only if the opinion vectors of agents i and j are orthogonal, a zero entry in the appraisal matrix is a very rare occurrence, as it will be confirmed by the numerical simulations at the end of the paper.

As we will see in the following, our model retains all the relevant features of the model investigated in [16], and it is simpler to analyse and implement.

Assumption 1. (No zero rows/columns). *In the following, we will steadily assume that* $\mathbf{Y}(0)$ *is devoid of zero rows/columns.*

Lemma 3 (No zero rows dynamics)**.** *If* $\mathbf{Y}(0) \in \mathbb{R}^{N\times m}$ *has no zero rows, then for every* $t \geq 0$ *the matrix* $\mathbf{Y}(t)$*, obtained from the model (6) corresponding to the initial condition* $\mathbf{Y}(0)$*, has no zero rows.*

Proof. Suppose, by contradiction, that this is not the case, and let $t_0 \geq 0$ be the smallest time instant such that $\mathbf{Y}(t_0)$ has no zero rows, but $\mathbf{Y}(t_0+1)$ has (at least) one zero row. It entails no loss of generality assuming that the first row of $\mathbf{Y}(t_0+1)$ becomes zero (if not we can always resort to a relabelling of the agents to reduce ourselves to this case). If we set $\mathbf{Y} := \mathbf{Y}(t_0)$, this means that $\mathbf{Y}$ has no zero rows, but

$$\mathbf{e}_1^\top \text{sgn}(\mathbf{Y}\mathbf{Y}^\top)\mathbf{Y} = \mathbf{0}^\top.$$

Set $\mathbf{z}^\top := \mathbf{e}_1^\top \text{sgn}(\mathbf{Y}\mathbf{Y}^\top) \in \{-1, 0, 1\}^{1\times N}$. We observe that since the first row of $\mathbf{Y}$ is not zero then the (1,1)-entry of $\mathbf{Y}\mathbf{Y}^\top$ is positive and hence the first entry of $\mathbf{z}$ is 1. The remaining ones belong to $\{-1, 0, 1\}$. We distinguish two cases: either all the other entries of $\mathbf{z}$ are zero (Case A) or there exist other nonzero entries in $\mathbf{z}$ (Case B), and in this latter case we can assume without loss of generality (if not, we can always permute the m topics, namely the m

columns of $\mathbf{Y}$, to make this possible) that

$$\mathbf{z}^\top = \begin{bmatrix} 1 & | & z_2 & \dots & z_r & | & 0 & \dots & 0 \end{bmatrix}, \quad \begin{matrix} z_i \in \{-1, 1\}, \\ i \in [2, r]. \end{matrix}$$

Condition $\mathbf{z}^\top \mathbf{Y} = \mathbf{0}^\top$ implies that the columns of $\mathbf{Y}$ are all orthogonal to the vector $\mathbf{z}$. In Case B this implies that $\mathbf{Y}$ can be expressed as

$$\mathbf{Y} = \left[\begin{array}{c|c} \begin{matrix} \mathbf{1}_{\mathbf{r-1}}^\top \\ \hline \Sigma \end{matrix} & \begin{matrix} \mathbf{0}^\top \\ \hline \mathbf{0}^\top \end{matrix} \\ \hline 0 & I_{N-r} \end{array}\right] \begin{bmatrix} C_a \\ C_b \end{bmatrix} \tag{7}$$

for some matrices $C_a \in \mathbb{R}^{(r-1)\times m}$ and $C_b \in \mathbb{R}^{(N-r)\times m}$, where $\Sigma := -\text{diag}\{z_2, \dots, z_r\}$. On the other hand, the vector $\mathbf{z}^\top$ and the matrix $\mathbf{Y}$ are related by the identity $\mathbf{z}^\top = \mathbf{e}_1^\top \text{sgn}(\mathbf{Y}\mathbf{Y}^\top) = \text{sgn}(\mathbf{e}_1^\top \mathbf{Y}\mathbf{Y}^\top)$, and hence it must be

$$\begin{aligned} &\begin{bmatrix} 1 & | & z_2 & \dots & z_r & | & 0 & \dots & 0 \end{bmatrix} \\ &\quad = \text{sgn}\left(\mathbf{1}_{r-1}^\top C_a \begin{bmatrix} C_a^\top & C_b^\top \end{bmatrix} \left[\begin{array}{c|c|c} \mathbf{1}_{\mathbf{r-1}} & \Sigma & 0 \\ 0 & 0 & I_{N-r} \end{array}\right] \right). \end{aligned}$$

This implies, in particular, that

$$\begin{bmatrix} z_2 & \dots & z_r \end{bmatrix} = \text{sgn}(\mathbf{1}_{r-1}^\top C_a C_a^\top \Sigma),$$

or, entrywise, keeping into account the definition of Σ:

$$z_i = -\text{sgn}(\mathbf{1}_{r-1}^\top C_a C_a^\top z_i \mathbf{e}_{i-1}), \quad \forall\, i \in [2, r].$$

This amounts to saying that

$$\text{sgn}(\mathbf{1}_{r-1}^\top C_a C_a^\top \mathbf{e}_{i-1}) = -1, \quad \forall\, i \in [2, r],$$

namely $\mathbf{1}_{r-1}^\top C_a C_a^\top \ll 0$, by this meaning that it is a vector with all negative entries. But this would imply $\|C_a^\top \mathbf{1}_{r-1}\|^2 = \mathbf{1}_{r-1}^\top C_a C_a^\top \mathbf{1}_{r-1} < 0$, which is clearly impossible.

We consider now Case A. If the only nonzero entry of $\mathbf{z}$ is the first one, then $\mathbf{Y}$ can be expressed as $\mathbf{Y} = \mathbf{W}C_0$, where $\mathbf{W} = [\mathbf{0}|I_{N-1}]^\top$ and C_0 is a real matrix of size $(N-1) \times m$. By resorting to the same reasoning as in Case B, condition $\mathbf{z}^\top = \mathbf{e}_1^\top \text{sgn}(\mathbf{Y}\mathbf{Y}^\top)$ becomes

$$\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} = \text{sgn}(\mathbf{e}_1 \mathbf{W} C_0 C_\mathbf{0}^\top \mathbf{W}^\top) = \text{sgn}(\mathbf{0}^\top),$$

which is impossible. Therefore it is not possible that there exists $t_0 \geq 0$ such that $\mathbf{Y}(t_0)$ has no zero rows, but $\mathbf{Y}(t_0 + 1)$ has (at least) one zero row. □

Based on the preliminary remarks and Lemma 3, we introduce the set [19]

$$\mathcal{S}_{nz-rows} := \{\mathbf{Y} \in \mathbb{R}^{N\times m} : \mathbf{e}_i^\top \mathbf{Y} \neq \mathbf{0}^\top, \forall\, i \in [1, N]\},$$

and in the following we will steadily assume that $\mathbf{Y}(0) \in \mathcal{S}_{nz-rows}$, and hence $\mathbf{Y}(t) \in \mathcal{S}_{nz-rows}$ for every $t \geq 0$. It is worth noticing that, differently from [16], we do not need to impose that $\mathbf{Y}(0) \in \mathcal{Y} := \{\mathbf{Y} : \mathbf{Y}(t) \in \mathcal{S}_{nz-rows} \forall t \geq 0\}$, since for our model it suffices to assume that $\mathbf{Y}(0) \in \mathcal{S}_{nz-rows}$ to guarantee that $\mathbf{Y}(t) \in \mathcal{S}_{nz-rows}$, $\forall t \geq 0$.

Note that, as a further consequence, for every $t \geq 0$, $\mathbf{X}(t+1) = \text{sgn}(\mathbf{Y}(t)\mathbf{Y}(t)^\top)$ is a symmetric matrix with unitary diagonal entries, and hence belongs to $\mathcal{S}_1^N$, $\forall t \geq 0$.

Remark 4. The case when there exists $t > 0$ such that the matrix $\mathbf{Y}(t)$ has a zero column, even if $\mathbf{Y}(0)$ has no zero columns, may arise, but it is a rare occurrence. This happens if and only if one of the columns of $\mathbf{Y}(t)$ belongs to the kernel of the matrix $\mathbf{X}(t+1) = sgn(\mathbf{Y}(t)\mathbf{Y}(t)^\top)$. This means that at the time t the column vector describing the opinions that the agents have on some specific topic is such that *for every agent* i the sum of the opinions of the agents trusted by i equals the sum of the opinions of the agents not trusted by agent i. Since the agents'opinions are arbitrary real numbers this case arises for a set of initial conditions $\mathbf{Y}(0)$ having zero measure.

An elementary example is represented by the case when $\mathbf{Y}(0) = \begin{bmatrix} 1 & \epsilon \\ 2 & -\epsilon \end{bmatrix}$, where ϵ is nonzero and sufficiently small. Correspondingly, we get $\mathbf{Y}(1) = \begin{bmatrix} 3/2 & 0 \\ 3/2 & 0 \end{bmatrix}$.

After having explored these preliminary aspects regarding agents that become indifferent to all issues, or issues that become irrelevant to all agents, we want to investigate the existence and structure of the equilibrium points for the model (2)- (3), when starting from initial opinion matrices $\mathbf{Y}(0)$ satisfying Assumption 1.

Definition 5 (Equilibrium point)**.** A pair $(\mathbf{Y}^*, \mathbf{X}^*)$ is an *equilibrium point* for the model (2)- (3) if

$$\mathbf{X}^* = \text{sgn}(\mathbf{Y}^*(\mathbf{Y}^*)^\top) \tag{8}$$

$$\mathbf{Y}^* = \frac{1}{N}\mathbf{X}^*\mathbf{Y}^*. \tag{9}$$

It is interesting to notice that the only possible nontrivial equilibrium points for the model are those that correspond to a structurally balanced configuration of the appraisal network $\mathcal{G}(\mathbf{X}^*)$. Moreover, the appraisal network is necessarily complete, namely each agent needs to express its appraisal towards all the other agents.

Proposition 6 (Equilibrium equivalence conditions)**.** *A pair* $(\mathbf{Y}^*, \mathbf{X}^*) \neq (\mathbf{0}, \mathbf{0})$ *is an equilibrium point for the model (2)- (3) if and only if*

i) $\mathbf{X}^* = \mathbf{p}\mathbf{p}^\top$, *for some* $\mathbf{p} \in \{-1, 1\}^N$*;*
ii) $\mathbf{Y}^* = \mathbf{p}\begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix}$, *for some* $a_i \in \mathbb{R}$, $\sum_{i=1}^m a_i^2 \neq 0$.

Proof. It is immediate to observe that if i) and ii) hold, then the identities (8) and (9) hold. Conversely, assume that the pair $(\mathbf{Y}^*, \mathbf{X}^*)$ is an equilibrium point. Then (9) holds, but this means that the nonzero columns of $\mathbf{Y}^*$ are eigenvectors of $\frac{1}{N}\mathbf{X}^*$ corresponding to the unitary eigenvalue. This means that $1 \in \sigma\left(\frac{1}{N}\mathbf{X}^*\right)$ and therefore, by Lemma 17 in the Appendix, i) holds. On the other hand, by replacing the matrix $\mathbf{X}^*$ in (9) with $\mathbf{p}\mathbf{p}^\top$, we obtain ii). □

Remark 7. The non-trivial equilibrium points of the model are *modulus consensus* configurations, see, e.g., [16]. This is also what happens for equilibrium points in [3] and [16]. Moreover, when the model converges to the non-trivial equlibrium configurations, the sign distribution of the opinions mirrors the network partition into factions.

As in Altafini's model [3], and as it will be clear in the following, the system dynamics either achieves modulus consensus (in a finite number of steps) or converges to zero (asymptotically).

Remark 8. This situation is different from the one that arises with the model investigated in [5], [16] and [18]. In [16] (see Remark 4 and section IV in [16]) the equilibrium points identified in the previous Proposition 6 are not the only possible ones. Indeed, for the model explored in [16] the case may occur that the matrix $\mathbf{X}^*$ at the equilibrium corresponds to a non connected graph, whose connected components however achieve structural balance. As we will see later (see Remark 14), if $\mathcal{G}(\mathbf{X}^*)$ becomes disconnected then both the opinion matrix and the appraisal matrix converge to zero. On the other hand, for the binary model in [18], convergence to a non-all-to-all structurally balanced network is also possible while the topological structure of the associated graph is time invariant. Also, under some conditions, convergence to "two-factions" structural balance is obtained in finite time. The signed formation game in [5] dynamically drives the network towards clustering balance.

We want now to show that the model we have proposed cannot exhibit periodic solutions and hence limit cycles. To prove this result we need a preliminary lemma, that will be useful also for the subsequent analysis.

Lemma 9 (Upper bounded opinion dynamics)**.** *For every $j \in [1, m]$ and every $t \geq 0$*
i)

$$\max_{i\in[1,N]} |Y_{ij}(t+1)| \leq \max_{i\in[1,N]} |Y_{ij}(t)|. \tag{10}$$

ii) Condition

$$\max_{i\in[1,N]} |Y_{ij}(t+1)| = \max_{i\in[1,N]} |Y_{ij}(t)| \neq 0$$

holds if and only if

(a) $\mathbf{Y}(t)\mathbf{e}_j = \mathbf{p}\cdot\mu_j,\ \exists\ \mathbf{p} \in \{-1, 1\}^N$ and $\mu_j > 0$; and

(b) once we set $h := \text{argmax}_{i\in[1,N]}|Y_{ij}(t+1)|$ then $\mathbf{e}_h^\top \mathbf{X}(t+1)$ has no zero entries and $\mathbf{e}_h^\top \mathbf{X}(t+1) = p_h \cdot \mathbf{p}^\top$.

Proof. i) From Eq. (3) it follows that

$$|Y_{ij}(t+1)| = |\frac{1}{N}\sum_{k=1}^{N} X_{ik}(t+1)Y_{kj}(t)| \leq \frac{1}{N}\sum_{k=1}^{N} |X_{ik}(t+1)||Y_{kj}(t)|$$
$$\leq \frac{1}{N}\sum_{k=1}^{N} |Y_{kj}(t)| \leq \frac{1}{N} N \max_k |Y_{kj}(t)| = \max_k |Y_{kj}(t)|,$$

and hence (10) holds. ii) Set $h := \text{argmax}_{i\in[1,N]}|Y_{ij}(t+1)|$. Then $|Y_{hj}(t+1)| = \max_{i\in[1,N]}|\ Y_{ij}(t+1)|$ coincides with $\max_{i\in[1,N]}|Y_{ij}(t)|$ if and only if

$$\sum_{\ell=1}^{N} |X_{h\ell}(t+1)||Y_{\ell j}(t)| = N\cdot \max_{i\in[1,N]} |Y_{ij}(t)|$$

and this is possible if and only if all the entries in the j-th column of $\mathbf{Y}(t)$ have the same absolute value $\mu_j > 0$ (and this leads to (a), for some suitable vector $\mathbf{p}$) and all the terms $X_{h\ell}(t+1)Y_{\ell j}(t)$, $\ell \in [1, N]$, have the same sign. But this latter condition means that $\mathbf{e}_h^\top \mathbf{X}(t+1)$ either coincides with $\mathbf{p}^\top$ or with its opposite, and since $X_{hh}(t+1) = 1$ this means that condition (b) holds. □

We are now in a position to prove the following result.

Proposition 10 (Aperiodicity in opinion dynamics)**.** *Suppose that there exist $\bar{t} \geq 0$, $T \geq 1$ and nonzero matrices $\tilde{\mathbf{Y}}_i \in \mathbb{R}^{N\times m}, i \in [1, T]$, such that*

$$\mathbf{Y}(\bar{t}+i) = \tilde{\mathbf{Y}}_i, i \in [1, T], \quad \text{and} \quad \mathbf{Y}(\bar{t}+T+1) = \tilde{\mathbf{Y}}_1,$$

namely from $\bar{t}+1$ onward the sequence of matrices $\{\mathbf{Y}(t)\}_{t\geq\bar{t}+1}$ becomes periodic of period T, then $T = 1$, namely the sequence becomes constant.

Proof. From Lemma 9, part i), we can claim that for every $j \in [1, m]$ and every $t \geq 0$

$$\max_{\ell\in[1,N]} |Y_{\ell j}(t+T+1)| \leq \max_{\ell\in[1,N]} |Y_{\ell j}(t+T)|$$
$$\leq \ldots \leq \max_{\ell\in[1,N]} |Y_{\ell j}(t+2)| \leq \max_{\ell\in[1,N]} |Y_{\ell j}(t+1)|.$$

But since for $t = \bar{t}$ we have $\mathbf{Y}(\bar{t}+T+1) = \mathbf{Y}(\bar{t}+1) = \tilde{\mathbf{Y}}_1$ and hence the two extremes in the previous sequence of inequalities coincide, it follows that all the symbols $\leq$ are equalities, namely

$$\max_{\ell\in[1,N]} |[\tilde{\mathbf{Y}}_i]_{\ell j}| = \mu_j > 0, \qquad \forall\ j \in [1, m],\ \forall\ i \in [1, T].$$

This also implies, see Lemma 9 part ii), that, for every non zero column in $\tilde{\mathbf{Y}}_i$,

$$\tilde{\mathbf{Y}}_i\mathbf{e}_j = \mathbf{p}_i\cdot\mu_j, \qquad \exists\ \mathbf{p}_i \in \{-1, 1\}^N, \mu_j > 0, \tag{11}$$

and that, for every $h \in [1, N]$, one has $\mathbf{e}_h^\top \text{sgn}(\tilde{\mathbf{Y}}_i\tilde{\mathbf{Y}}_i^\top) = [\mathbf{p}_i]_h \cdot \mathbf{p}_i^\top$. This implies that for every $i \in [1, T]$

$$\text{sgn}([\tilde{\mathbf{Y}}_i\tilde{\mathbf{Y}}_i^\top]) = \mathbf{p}_i\mathbf{p}_i^\top, \qquad \exists\ \mathbf{p}_i \in \{-1, 1\}^{N\times N}.$$

Consequently[2]

$$\tilde{\mathbf{Y}}_{(i+1\text{mod}T)} = \frac{1}{N}\text{sgn}([\tilde{\mathbf{Y}}_i\tilde{\mathbf{Y}}_i^\top])\tilde{\mathbf{Y}}_i = \frac{1}{N}\mathbf{p}_i\mathbf{p}_i^\top\tilde{\mathbf{Y}}_i. \tag{12}$$

So, by comparing (11) and (12) one gets that every matrix $\tilde{\mathbf{Y}}_i, i \in [1, T]$, takes the form

$$\tilde{\mathbf{Y}}_i = \mathbf{p}_i\begin{bmatrix} a_1^{(i)} & \ldots & a_m^{(i)} \end{bmatrix}, \qquad \exists\ \mathbf{p}_i \in \{-1, 1\}^N, a_k^{(i)} \in \mathbb{R},$$

but this also implies that

$$\tilde{\mathbf{Y}}_{(i+1\text{mod}T)} = \frac{1}{N}\text{sgn}([\tilde{\mathbf{Y}}_i\tilde{\mathbf{Y}}_i^\top])$$
$$\tilde{\mathbf{Y}}_i = \frac{1}{N}\text{sgn}(\mathbf{p}_i\mathbf{p}_i^\top\cdot\sum_k [a_k^{(i)}]^2)\mathbf{p}_i\begin{bmatrix} a_1^{(i)} & \ldots & a_m^{(i)} \end{bmatrix}$$
$$= \frac{1}{N}\mathbf{p}_i\mathbf{p}_i^\top\mathbf{p}_i\begin{bmatrix} a_1^{(i)} & \ldots & a_m^{(i)} \end{bmatrix} = \mathbf{p}_i\begin{bmatrix} a_1^{(i)} & \ldots & a_m^{(i)} \end{bmatrix} = \tilde{\mathbf{Y}}_i.$$

So, all matrices $\tilde{\mathbf{Y}}_i$ coincide. □

3. Convergence to an equilibrium in a finite number of steps

We want to explore under what conditions the equilibrium can be reached in a finite number of steps. It is easy to see that if there exists a time instant $t_0 \geq 0$ such that $\mathbf{Y}(t_0+1) = \mathbf{Y}(t_0) \neq 0$ then $\mathbf{Y}(t) = \mathbf{Y}(t_0) =: \mathbf{Y}^*$ for every $t \geq t_0$. Consequently, also $\mathbf{X}(t)$ becomes constant starting at $t = t_0 + 1$, and it coincides with $\mathbf{X}^* := \text{sgn}(\mathbf{Y}(t_0)\mathbf{Y}(t_0)^\top)$.

However, the converse is not true: if the appraisal matrix becomes constant at some time $t_0 \geq 0$, the opinion matrix $\mathbf{Y}(t)$ can still keep evolving for $t \geq t_0$. This situation is illustrated in Example 11, below.

As a matter of fact, if there exists a time instant $t_0 \geq 0$ such that $\mathbf{X}(t) = \mathbf{X}^*, \forall t \geq t_0$, we can only claim that $\mathbf{Y}(t+1) = \frac{1}{N}\mathbf{X}^*\mathbf{Y}(t)$. Equivalently, if we denote by $\mathbf{y}_j(t)$, the j-th column of the matrix $\mathbf{Y}(t)$, then the dynamics expressed by Eq. (2) decomposes into m linear time invariant systems of the form

$$\mathbf{y}_j(t+1) = \frac{1}{N}\mathbf{X}^*\mathbf{y}_j(t),\ \forall\ j \in [1, m]. \tag{13}$$

As $\mathbf{X}^* \in \mathcal{S}_1^N$, the matrix $\frac{\mathbf{X}^*}{N}$ is symmetric and hence diagonalizable. Moreover, by Gershgorin Circle theorem, all its (real) eigenvalues $\lambda_i, i \in [1, N]$, satisfy

$$|\lambda_i - \frac{1}{N}| \leq \frac{N-1}{N} \iff -\frac{N-2}{N} \leq \lambda_i \leq 1,\ \forall i \in [1, N].$$

As a consequence, two cases may arise. The first case is the one depicted in Example 11, namely the case when the systems in (13) are asymptotically stable, which means that $\mathbf{Y}(t)$ asymptotically converges to 0 (and hence $\lim_{t\to+\infty}\mathbf{X}(t) = 0 \neq \mathbf{X}^*$).

Example 11. Let us consider the case $N = m = 3$, with

$$\mathbf{Y}(0) = \begin{bmatrix} 1.41 & -1.21 & 0.49 \\ 1.42 & 0.72 & 1.03 \\ 0.67 & 1.63 & 0.73 \end{bmatrix}.$$

It turns out that $\forall t \geq 1$

$$\mathbf{X}(t) = \mathbf{X}^* = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

and $\sigma(1/3\cdot\mathbf{X}^*) = (-1/3, 2/3, 2/3)$, and indeed for $t \geq 14$ we have $Y_{ij}(t) = o(10^{-2}), \forall i, j \in [1, 3]$.

The second possible situation is when $\frac{\mathbf{X}^*}{N}$ is simply (but not asymptotically) stable. This amounts to saying that 1 is a (simple)

[2] The expression $i+1\text{mod}T$ means the remainder of $i+1$ when divided by T.

eigenvalue of $\frac{\mathbf{X}^*}{N}$, and hence by Lemma 17, $\mathbf{X}^*$ takes the form $\mathbf{X}^* = \mathbf{p}\mathbf{p}^\top$, $\exists\ \mathbf{p} \in \{-1, 1\}^N$. In this case, the convergence is not asymptotic but instantaneous. In fact, it is sufficient that $\frac{1}{N}\mathbf{X}(t_0)$ becomes simply (but not asymptotically) stable at a single time instant, to ensure the instantaneous convergence of $\mathbf{Y}(t)$ to an equilibrium condition.

Proposition 12 (Equilibrium points characterization)**.** *If there exists $t_0 > 0$ such that $\frac{1}{N}\mathbf{X}(t_0)$, with $\mathbf{X}(t_0) \in \mathcal{S}_1^N$, is simply (but not asymptotically) stable, then $(\mathbf{X}^*, \mathbf{Y}^*) := (\mathbf{X}(t_0), \frac{1}{N}\mathbf{X}(t_0)\mathbf{Y}(t_0 - 1))$ is an equilibrium point.*

Proof. By Lemma 17 in the Appendix, we know that if $\frac{1}{N}\mathbf{X}(t_0) \in \mathcal{S}_1^N$ is simply stable or, equivalently, $1 \in \sigma(\frac{1}{N}\mathbf{X}(t_0))$, then there exists a vector $\mathbf{p} \in \{-1, 1\}^N$ such that $\mathbf{X}(t_0) = \mathbf{p}\mathbf{p}^\top$. On the other hand, if $\mathbf{X}(t_0) = \mathbf{p}\mathbf{p}^\top$, then

$$\mathbf{Y}(t_0) = \frac{1}{N}\mathbf{p}\mathbf{p}^\top\mathbf{Y}(t_0 - 1) = \mathbf{p}[a_1, \dots, a_m],$$

where

$$[a_1, \dots, a_m] := \frac{1}{N}\mathbf{p}^\top\mathbf{Y}(t_0 - 1).$$

Therefore $(\mathbf{X}^*, \mathbf{Y}^*) := (\mathbf{X}(t_0), \frac{1}{N}\mathbf{X}(t_0)\mathbf{Y}(t_0 - 1))$ is an equilibrium point. □

Remark 13. If $m = 1$ the model reaches the equilibrium in one step. When so, in fact $\mathbf{X}(1) = \text{sgn}(\mathbf{Y}(0)\mathbf{Y}^\top(0)) = \mathbf{p}\mathbf{p}^\top$, where $\mathbf{p} := \text{sgn}(\mathbf{Y}(0))$. Numerical simulations at the end of the paper will show that, when $m > 1$, namely multiple topics are considered, convergence to structural balance is almost surely guaranteed, and it occurs in a rather small number of steps even for medium size networks (e.g. $N = 100$).

Remark 14. Gershgorin Circle theorem also allows to say that if $\mathbf{X}^*$ is the adjacency matrix of a disconnected graph, all the eigenvalues of the matrix $\frac{\mathbf{X}^*}{N}$ lie in the circle of the complex plane of center the origin and radius $\frac{N-1}{N}$ (or smaller), and hence $\frac{\mathbf{X}^*}{N}$ is necessarily an asymptotically stable matrix.

Theorem 15 summarizes the main results of this section.

Theorem 15 (Main theorem)**.** *The following conditions are equivalent*

i) *there exists a time instant $t_0 \geq 0$ such that $1 \in \sigma(\frac{1}{N}\mathbf{X}(t_0))$;*
ii) *there exists a time instant $t_0 \geq 0$ such that $\mathbf{Y}(t_0) = \mathbf{Y}(t_0 + 1)$;*
iii) *the opinion-appraisal dynamic model (2)- (3) converges in finite time to an equilibrium $(\mathbf{X}^*, \mathbf{Y}^*)$;*
iv) *the opinion-appraisal dynamic model (2)- (3) converges in finite time to an equilibrium $(\mathbf{X}^*, \mathbf{Y}^*)$, with $\mathbf{X}^* = \mathbf{p}\mathbf{p}^\top$ and $\mathbf{Y}^* = \mathbf{p}[a_1, \dots, a_m]$, $\exists\ \mathbf{p} \in \{-1, 1\}^N$, and $a_i \in \mathbb{R}, i \in [1, m]$, with $\sum_{i=1}^m a_i^2 \neq 0$.*

Proof. iv) ⇔ iii) follows from Proposition 6. iii) ⇒ ii) is obvious, while the converse has been commented upon at the beginning of the section. i) ⇒ iv) follows from Proposition 12, while iv) ⇒ i) is obvious. □

4. Long term behavior

In the previous section, we have investigated what happens if either $\mathbf{Y}(t)$ or $\mathbf{X}(t)$ become constant starting at some time instant. In the former case the overall system (2)- (3) reaches the equilibrium in a finite number of steps. In the latter case a nontrivial equilibrium is reached if and only if $\mathbf{X}(t)$ at some point becomes structurally balanced. Differently the opinion matrix asymptotically converges to zero. We want to investigate now if a nontrivial equilibrium can be reached asymptotically, but not in a finite number of steps. An immediate consequence of the analysis of the previous section is that if the sequence of appraisal matrices $\{\mathbf{X}(t)\}_{t\geq 1}$ does not converge in a finite number of steps then $\frac{\mathbf{X}(t)}{N}$ is an asymptotically stable matrix for every $t \geq 1$. This means that if we define the set

$$\mathcal{S}_{stable} := \mathcal{S}_1^N \setminus \{\mathbf{X} \in \mathcal{S}_1^N : \mathbf{X} = \mathbf{p}\mathbf{p}^\top, \exists\, \mathbf{p} \in \{-1, 1\}^N\}, \tag{14}$$

then $\mathbf{X}(t) \in \mathcal{S}_{stable}$ for every $t \geq 1$.

Proposition 16 (Zero vanishing condition)**.** *If for every $t \geq 0$, $\mathbf{X}(t) \in \mathcal{S}_{stable}$, then $\lim_{t\to+\infty} \mathbf{Y}(t) = 0$.*

Proof. For every $j \in [1, m]$, let us define $\mu_j(t) := \max_{i\in[1,N]} |Y_{ij}(t)|$, and let us introduce the (generalized) Lyapunov function for the system in Eq. (6), $V : \mathbb{R}^{N\times m} \to \mathbb{R}$, defined as

$$V(\mathbf{Y}(t)) := \sum_{j=1}^m \mu_j(t).$$

We notice that $V(\mathbf{Y}) \geq 0$, $\forall \mathbf{Y} \in \mathbb{R}^{N\times m}$ and that $V(\mathbf{Y}) = 0$ if and only if $\mathbf{Y} = 0$. Define $\Delta_2 V(\mathbf{Y}(t)) := V(\mathbf{Y}(t + 2)) - V(\mathbf{Y}(t))$. We want to prove that $\Delta_2 V(\mathbf{Y}(t)) < 0$, $\forall t \geq 0$.

By Lemma 9 it immediately follows that $\Delta_2 V(\mathbf{Y}(t)) = \sum_{j=1}^m \mu_j(t + 2) - \mu_j(t) \leq 0$. We show now that there is not a time instant $t_0 \geq 0$ such that $\Delta_2 V(\mathbf{Y}(t)) = 0$. If this were the case, in fact, this would mean that $\forall j \in [1, m]$, $\mu_j(t_0 + 2) = \mu_j(t_0)$ and therefore $\mu_j(t_0 + 2) = \mu_j(t_0 + 1) = \mu_j(t_0) =: \mu_j$. As a consequence of Lemma 9 we deduce that

a) $\mathbf{Y}(t_0)\mathbf{e}_j = \mathbf{p}_j(t_0)\mu_j$, $\exists\ \mathbf{p}_j(t_0) \in \{-1, 1\}^N$, $\mathbf{Y}(t_0 + 1)\mathbf{e}_j = \mathbf{p}_j(t_0 + 1)\mu_j$, $\exists\ \mathbf{p}_j(t_0 + 1) \in \{-1, 1\}^N$,
b) $\forall h \in [1, N]$, $\mathbf{e}_h^T\mathbf{X}(t_0 + 1) = p_h\mathbf{p}_j(t_0)^\top$,

from which it follows that $\mathbf{X}(t_0 + 1) = \mathbf{p}_j(t_0)\mathbf{p}_j(t_0)^\top$ for every $j \in [1, m]$. But then $\mathbf{X}(t_0 + 1) = \mathbf{p}(t_0)\mathbf{p}(t_0)^\top$ with $\mathbf{p}(t_0) = \pm\mathbf{p}_j(t_0)$, $\forall j \in [1, m]$, that means that $\mathbf{X}(t_0 + 1)$ is structurally balanced and hence it does not belong to $\mathcal{S}_{stable}$, thus contradicting the hypotheses. Consequently, it must be $\Delta_2 V(\mathbf{Y}(t)) < 0$, $\forall t \geq 0$. Finally, by defining $\Delta_1 V(\mathbf{Y}(t)) := V(\mathbf{Y}(t + 1)) - V(\mathbf{Y}(t))$ we get that

$$\Delta_2 V(\mathbf{Y}(t)) + \Delta_1 V(\mathbf{Y}(t)) < 0, \quad \forall t \geq 0,$$

so the thesis follows as a direct consequence of Theorem 2.1 in [2]. □

Summarizing, Theorem 15 and Proposition 16 show that either there exists a time instant t_0 such that $\forall t \geq t_0$, $\mathbf{X}(t) = \mathbf{p}\mathbf{p}^\top$ and consequently $\mathbf{Y}(t) = \mathbf{p}[a_1, a_2, \dots, a_m]$, $a_i \in \mathbb{R}$, $\sum_i a_i^2 \neq 0$, otherwise, if a time instant t such that $\mathbf{X}(t)$ reaches the structural balance does not exist, then $\mathbf{Y}(t)$ converges to zero as time goes to infinity.

5. Simulations

In this section we show the outcome of Monte Carlo simulations in order to validate the convergence properties of the model. Fig. 1 shows how the average number of iterations needed in order to reach a structural balanced configuration over the total number of 30000 simulations varies as a function of the number of topics $m \in [1, 10]$, for networks involving $N = 9, 20, 100$ agents. Simulations are based on initial conditions $\mathbf{Y}(0)$ with entries independently drawn from a Gaussian random variable with zero mean and standard deviation $\sigma = 10$, namely $Y_{ij}(0) \sim \mathcal{N}(0, 100)$. It turns out that, in accordance with the Chernoff bound, by running 30000 simulations, the estimated probability $\hat{p}$ to reach a structurally balanced configuration is equal to 1 with accuracy $\epsilon = 0.01$ and confidence level $1 - \delta = 0.99$, namely $P(|\hat{p} - p| \leq \epsilon) \geq 1 - \delta$, for the case of $N = 20, 100$ agents, regardless of the number of topics taken into account while $\hat{p}$ is greater than or equal to 0.98 for all $m \in [1, 10]$, for $N = 9$, with the same accuracy and confidence interval.

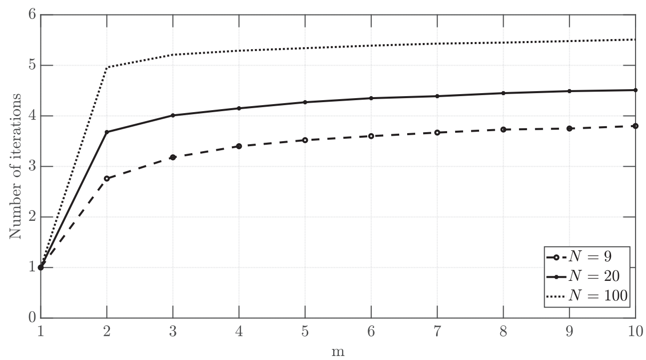

Fig. 1. Average number of iterations, over 30000 simulations, needed in order to reach a structural balance configurations for the cases $N = 9, 20, 100$ and $m \in [1, 10]$.

6. Conclusion

In this paper we have proposed a modified version of Liu et al. model [16] for the interplay between homophily-based appraisal dynamics and influence-based opinion dynamics. In order to update the agents' opinions on a numbers of issues, only the signs (and not the values) of the agents' mutual appraisals are used. This simplified model retains all the main characteristics of the original model, is simpler to analyse and implement, leads to the same kind of nontrivial structurally balanced equilibria as in [16], but rules out nontrivial equilibria that correspond to disconnected socially balanced networks. Furthermore, nontrivial equlibria can always be reached in a finite number of steps, while the case when all opinions and appraisals converge to zero corresponds to sets of initial conditions of zero measure.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Lemma 17 (Rank-one matrices with special structures)**.** *Given a matrix* $\mathbf{M} \in S_1^N$, *if* $1 \in \sigma\left(\frac{1}{N}\mathbf{M}\right)$, *then* $\mathbf{M} = \mathbf{p}\mathbf{p}^\top$ *for some* $\mathbf{p} \in \{-1, 1\}^N$, *and hence* $\mathbf{M}$ *has no zero entries and* $\sigma(\mathbf{M}) = (0, \dots, 0, 1)$.

Proof. Let $\mathbf{v} := \begin{bmatrix} v_1 & v_2 & \dots & v_N \end{bmatrix}^\top \in \mathbb{R}^N, \mathbf{v} \neq 0$, be an eigenvector of $\frac{1}{N}\mathbf{M}$ corresponding to the unitary eigenvalue, or equivalently of $\mathbf{M}$ corresponding to N. Then $\mathbf{M}\mathbf{v} = N\mathbf{v}$. Let $h := \text{argmax}_{i \in [1,N]} |v_i|$. Then condition

$$Nv_h = \sum_{i=1}^{N} M_{hi} v_i = v_h + \sum_{\substack{i=1 \\ i \neq h}}^{N} M_{hi} v_i$$

holds if and only if (a) $|v_i| = |v_h|$ for every $i \in [1, N]$; (b) $M_{hi} \neq 0$ for every $i \in [1, N]$, and $\text{sgn}(M_{hi})\text{sgn}(v_i) = \text{sgn}(v_h)$.

This implies that $\mathbf{v} = \mathbf{p}m$ for some $\mathbf{p} \in \{-1, 1\}^N$ and some $m > 0$ and $\mathbf{e}_h^\top \mathbf{M} = \text{sgn}(v_h)\mathbf{p}^\top = p_h \mathbf{p}^\top$.

On the other hand, since condition (a) holds, this means that every index $j \in [1, N]$ is $\text{argmax}_{i \in [1,N]} |v_i|$, and hence all the rows of $\mathbf{M}$ satisfy $\mathbf{e}_i^\top \mathbf{M} = p_i \mathbf{p}^\top$. This implies that $\mathbf{M} = \mathbf{p}\mathbf{p}^\top$, and the rest immediately follows. □

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