Critical properties of Heider balance on multiplex networks

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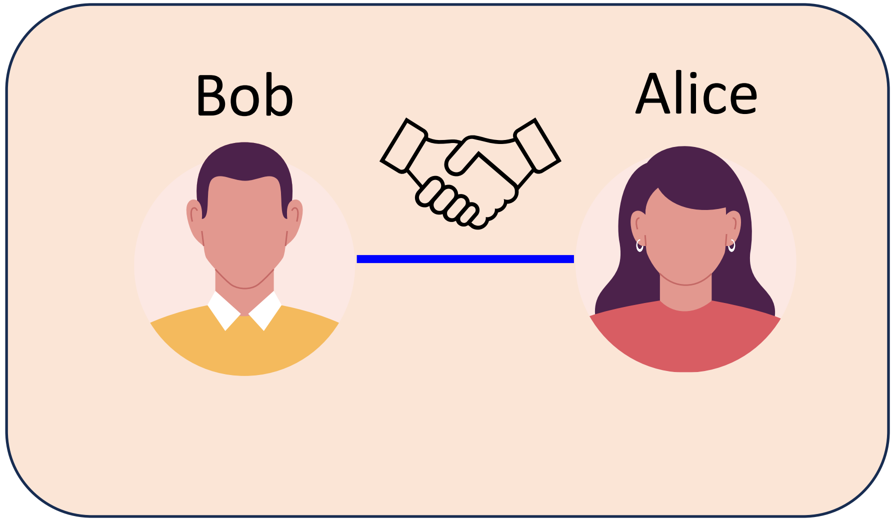
Introduction

- Relations can be friendly or Hostile



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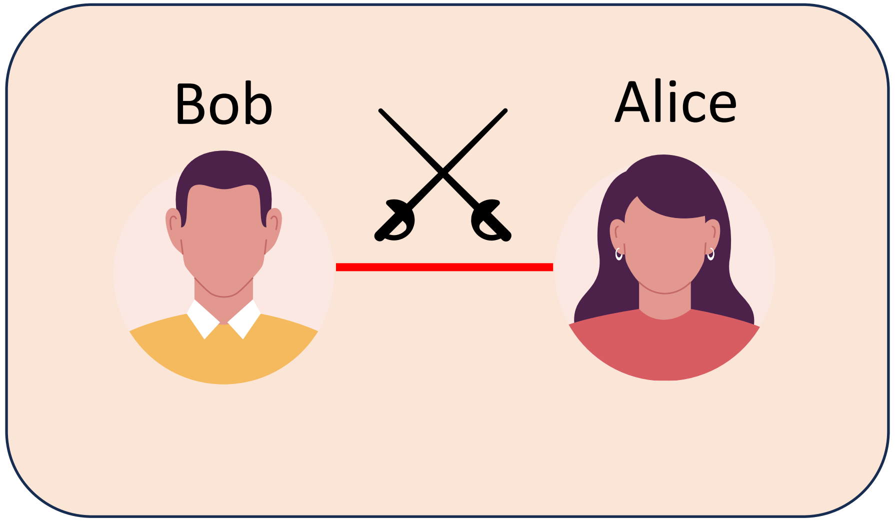

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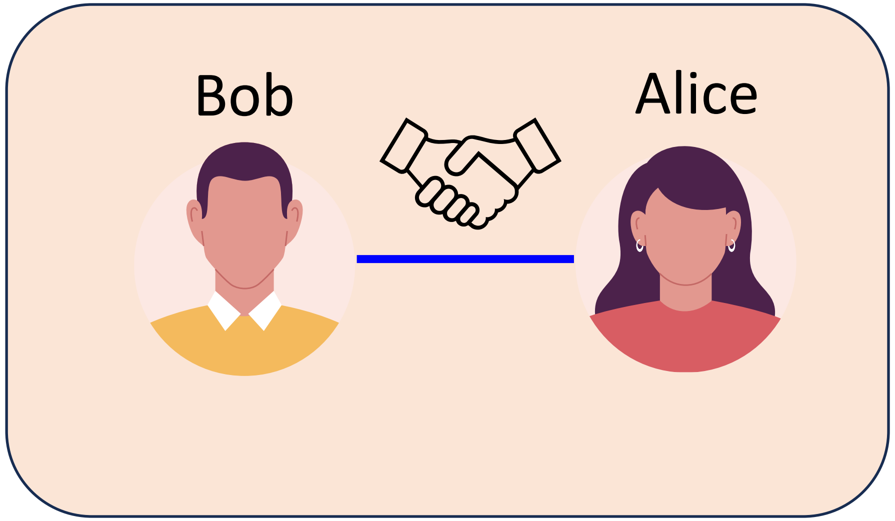
- Relations can be Positive or Negative



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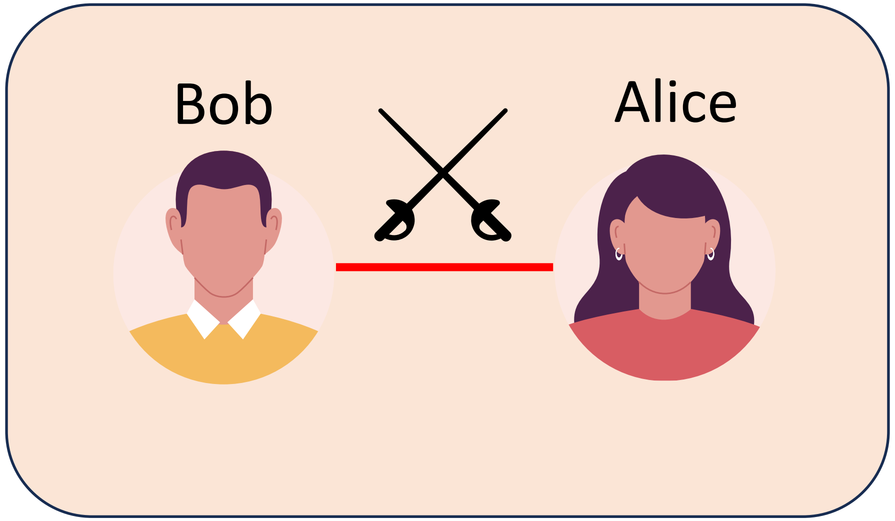

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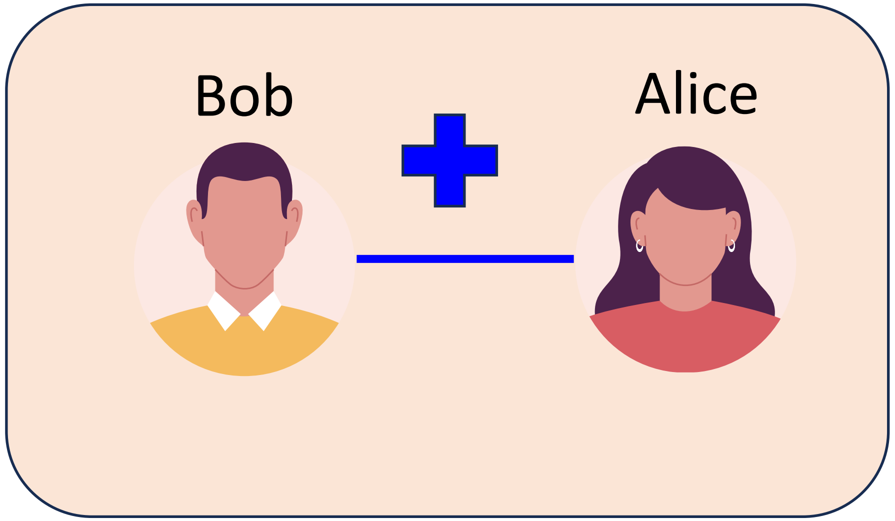

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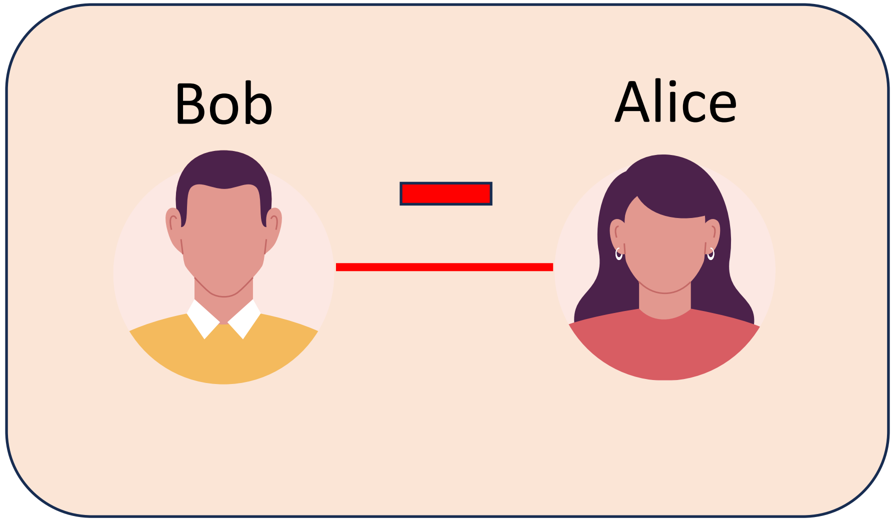

Introduction

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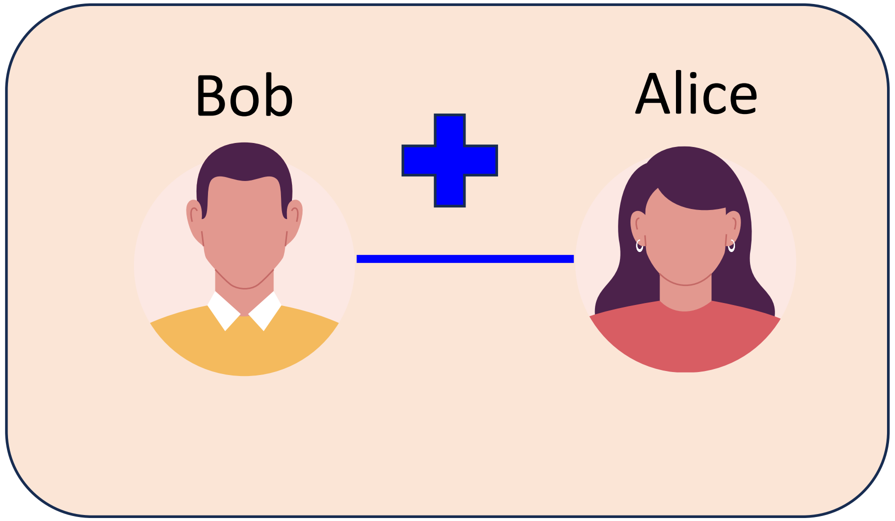
Signed Networks



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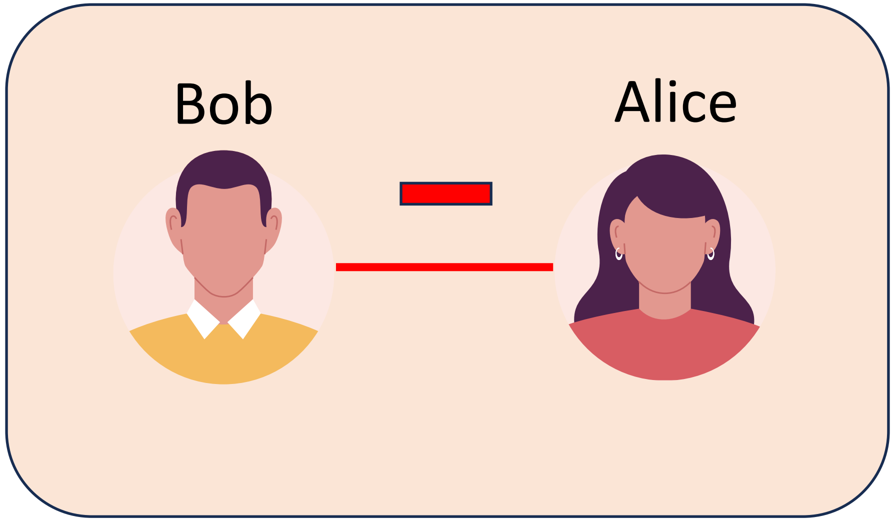
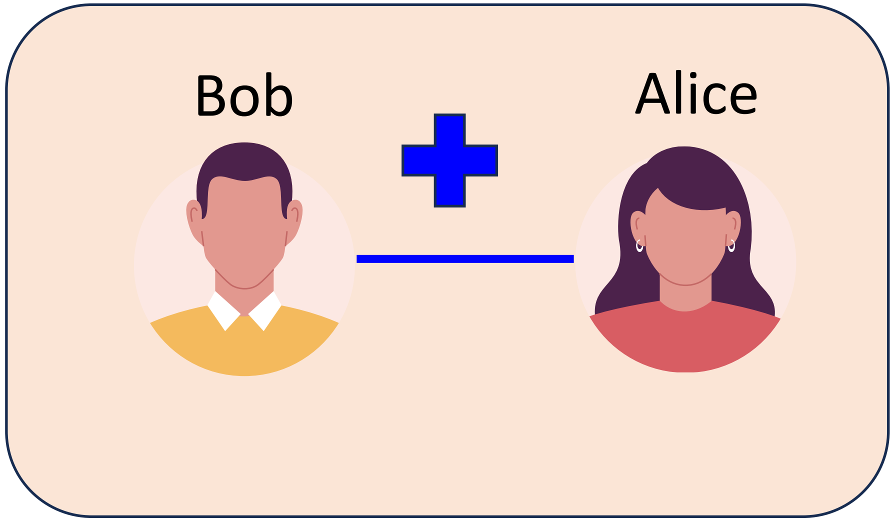


Introduction

- People usually have a consistency in their relationships.
- **Example**: Consistency in behavior at work and outside work.



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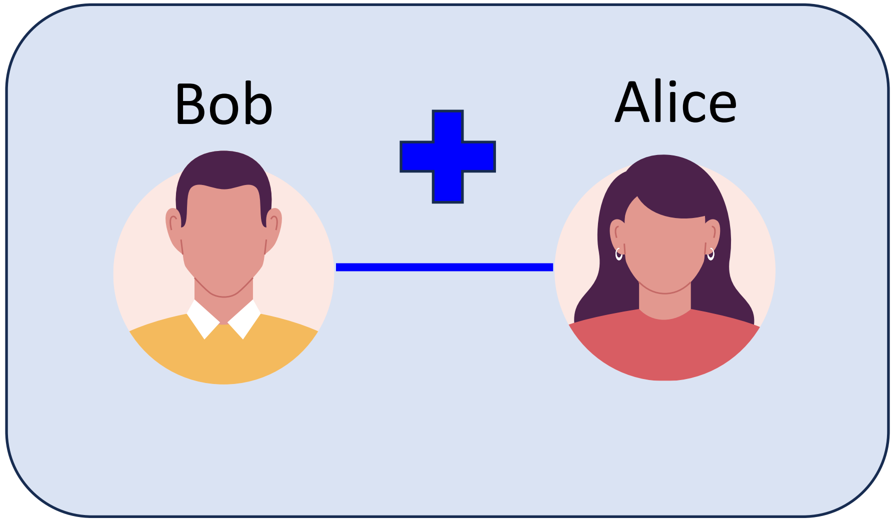
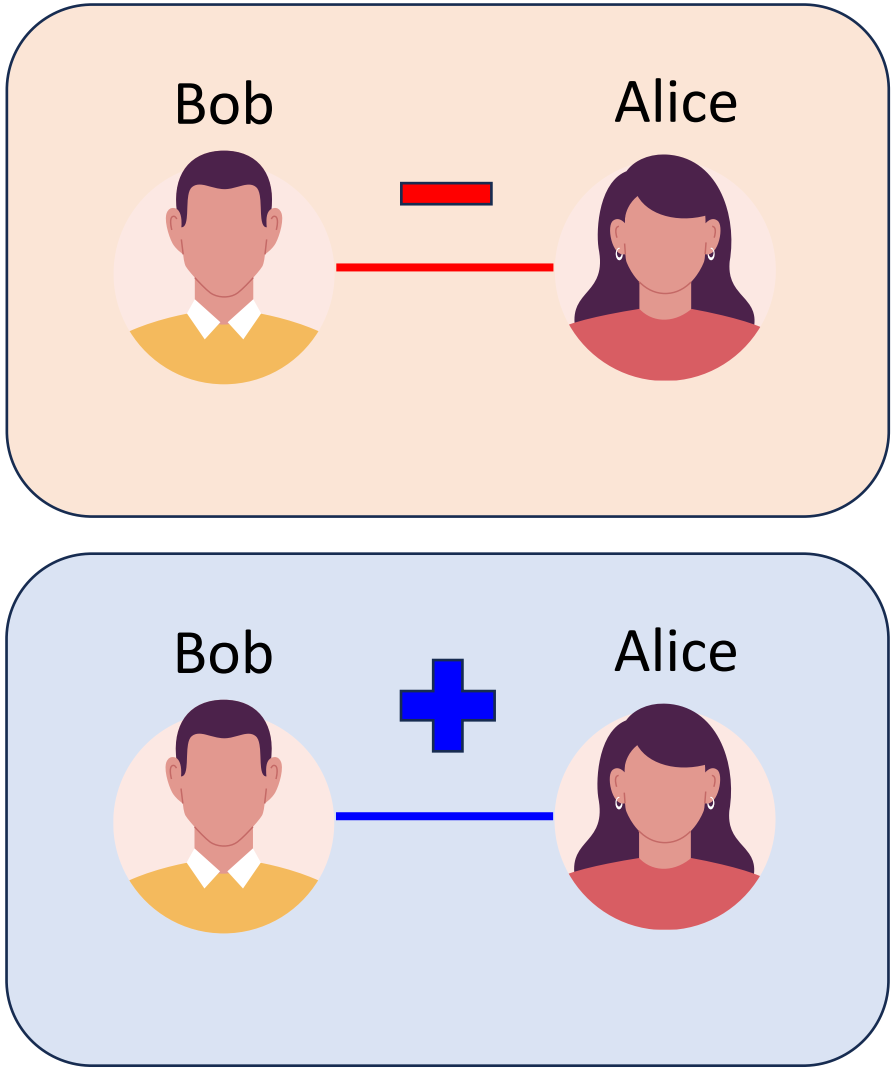


Introduction

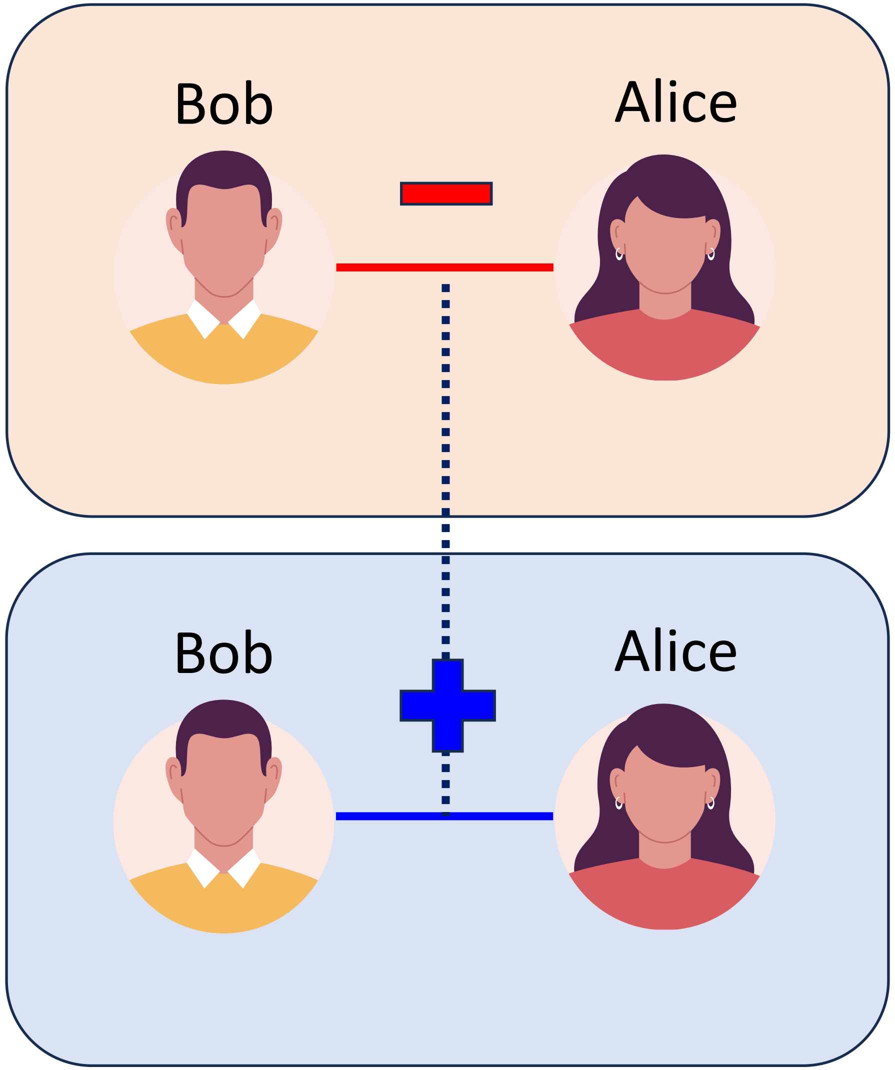
- Sports club and work relationship
- Variations in consistency lead to a cognitive dissonance



Introduction

- Officially friends and privately enemies?
- Variations in consistency lead to a cognitive dissonance

Interactions between different relations linking the same pair of individuals



Introduction

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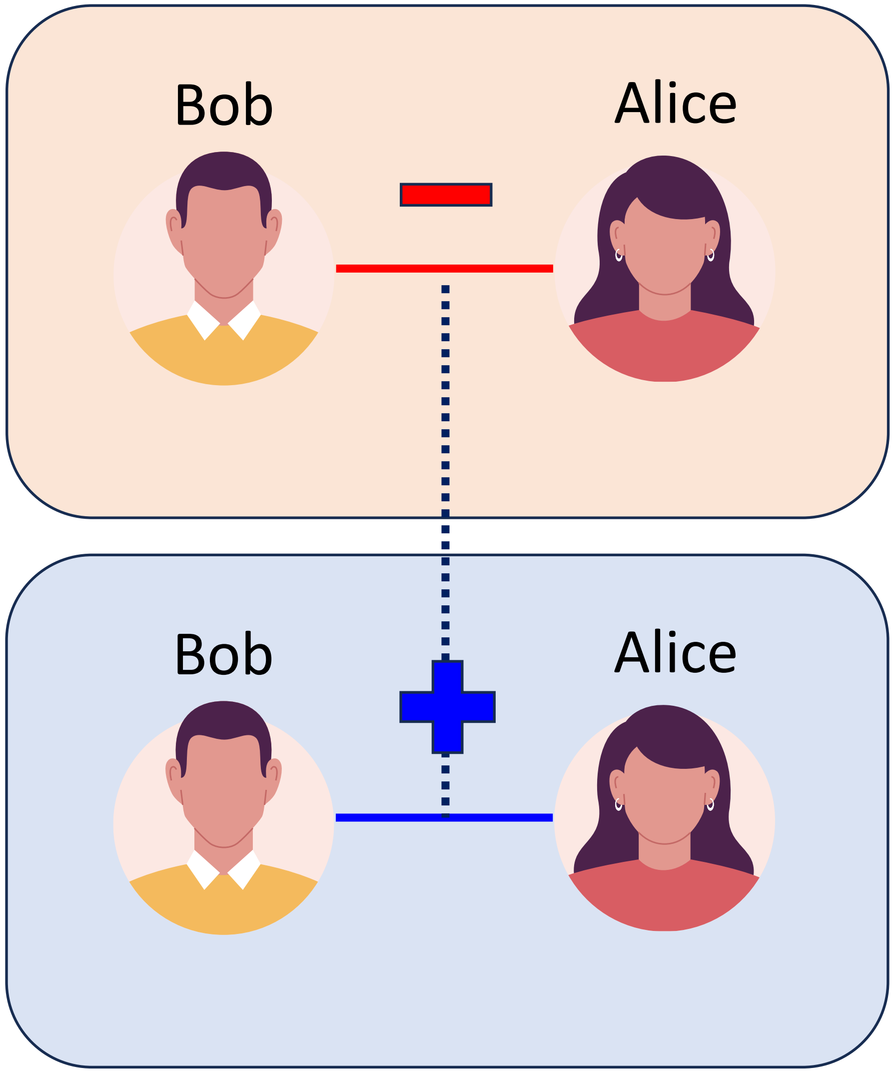
- Variations in consistency lead to a cognitive dissonance

Interactions between different relations linking the same pair of individuals

Interactions is for the relations and not between the people

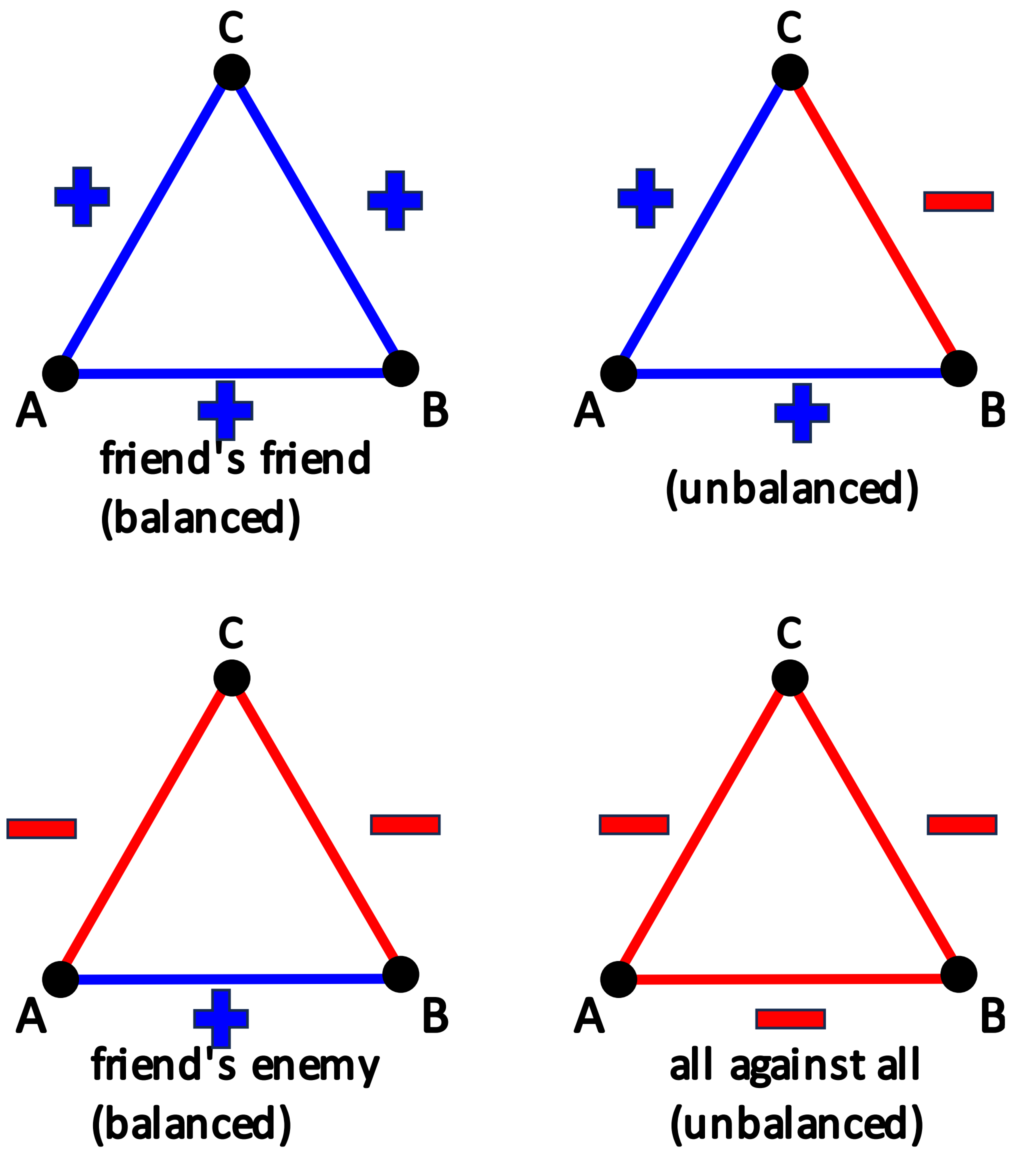


Outline

Social science Lens		Physics Lens
Studying different relationships to understand their impact on social balance.	Heider Balance to multiplex networks	How multiplex network effect the Heider balance
What happens to the relationship layers	Simulations	Monte Carlo simulation assesses how changes in one layer affect network balance.
Two layers in different relations.	Duplex network with contrasting relationship layers.	A duplex network in different states
Missing connection	HB with missing links	Erdös Rényi graph

Heider Balance theory

- Friend of my friend is my friend
- Enemy of my enemy is my friend
- Friend of my enemy is my enemy
- Enemy of my friend is my enemy



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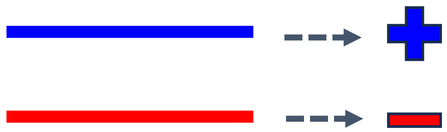
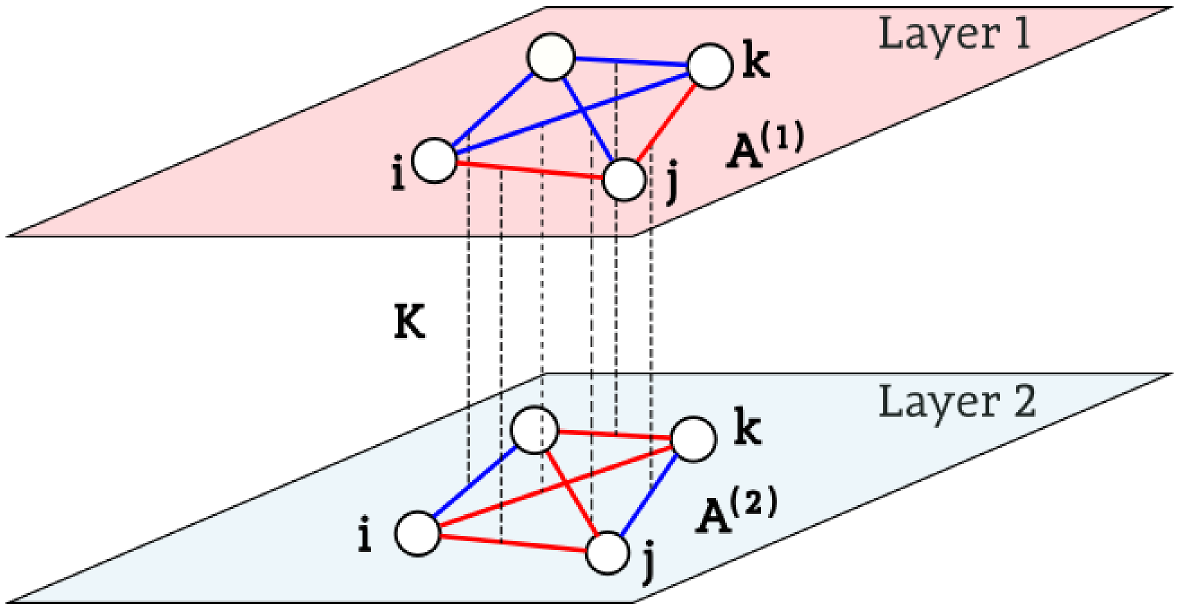


Assumptions:
- $x_{ij}^{(\alpha)} = \pm 1$
- Each layer corresponds to a different type of relationship
- Interlayer interactions exist between different relations for the same pair of agents

Hamiltonian (Utility Potential):

$$H = -\sum_{\alpha=1}^{L} A^{(\alpha)} \sum_{i>j>k} x_{ij}^{(\alpha)} x_{jk}^{(\alpha)} x_{ki}^{(\alpha)} - K \sum_{\alpha>\beta} \sum_{i>j} x_{ij}^{(\alpha)} x_{ij}^{(\beta)}$$



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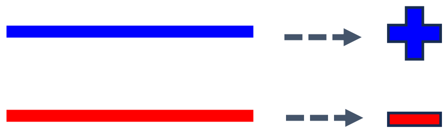

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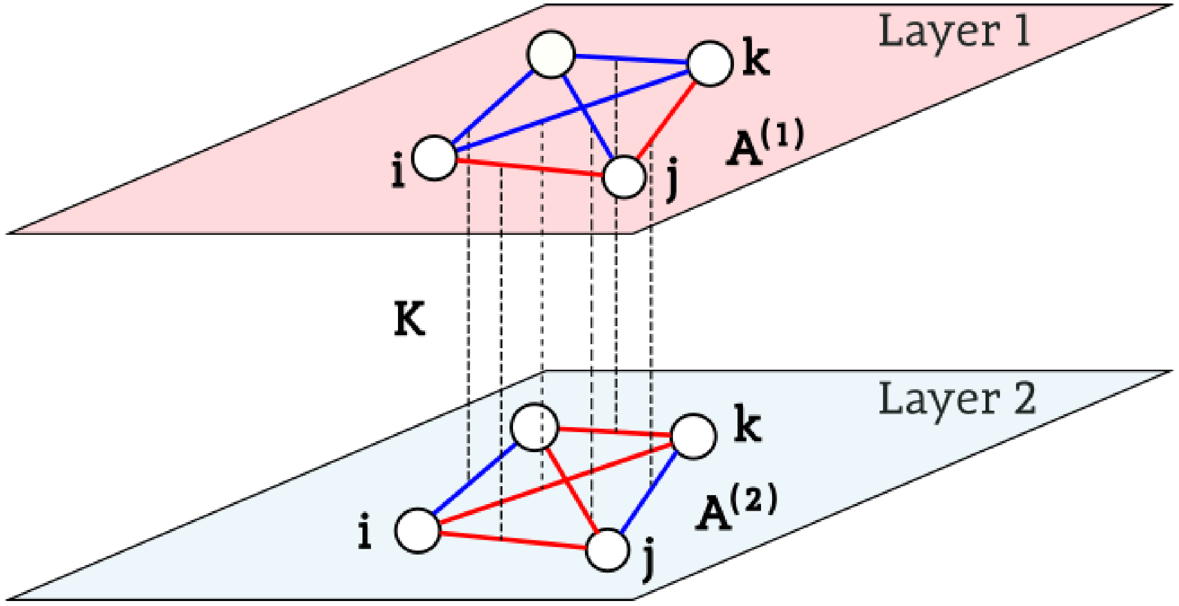
Heider interactions
(in the layer)



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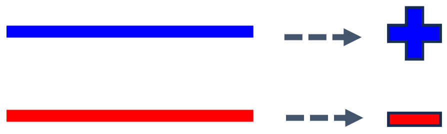

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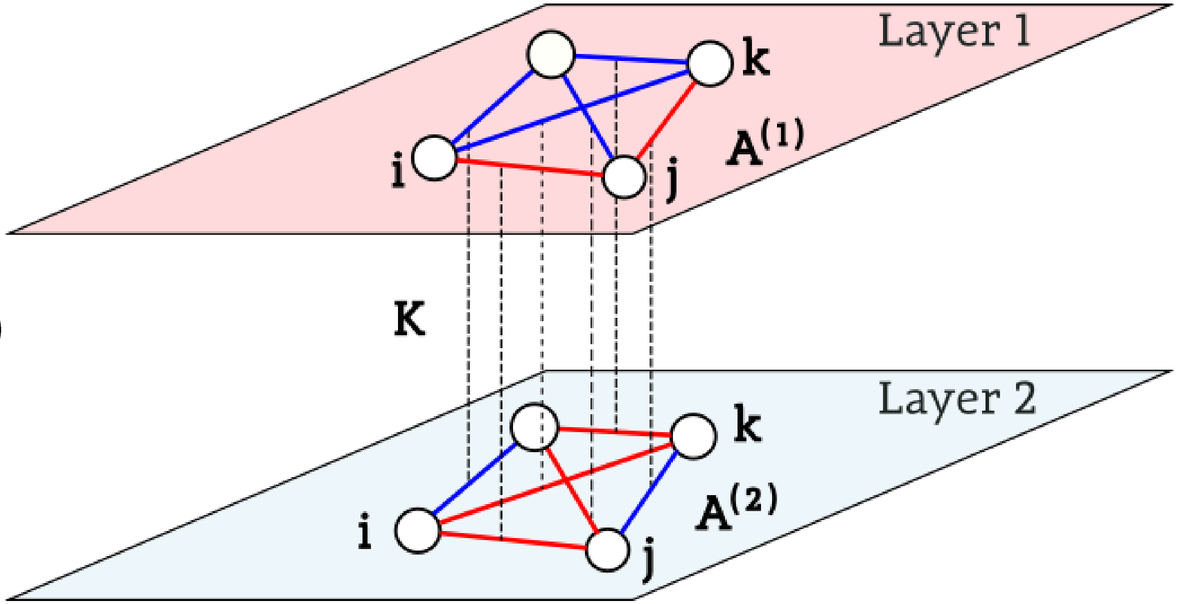
Heider interactions (in the layer)

Ising Interactions (between the layer)



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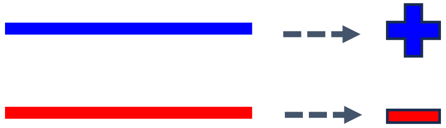

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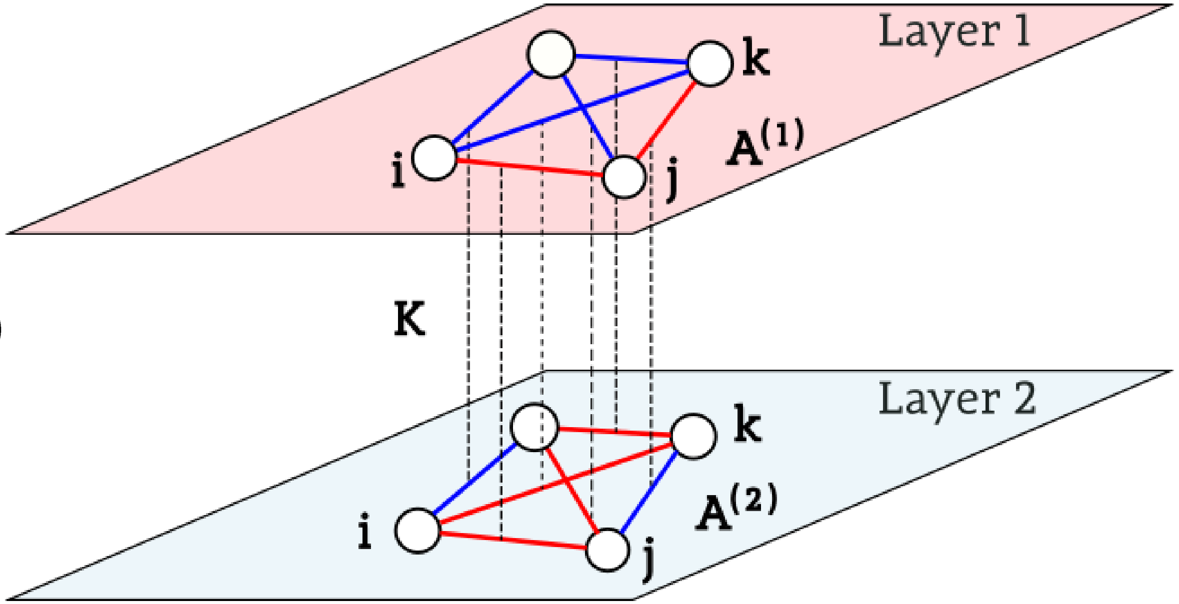
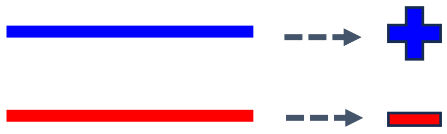


$\propto = 1, 2 \dots, L$ indicates the layer
A = Heider interaction strength
K = Ising Interaction strength

Model



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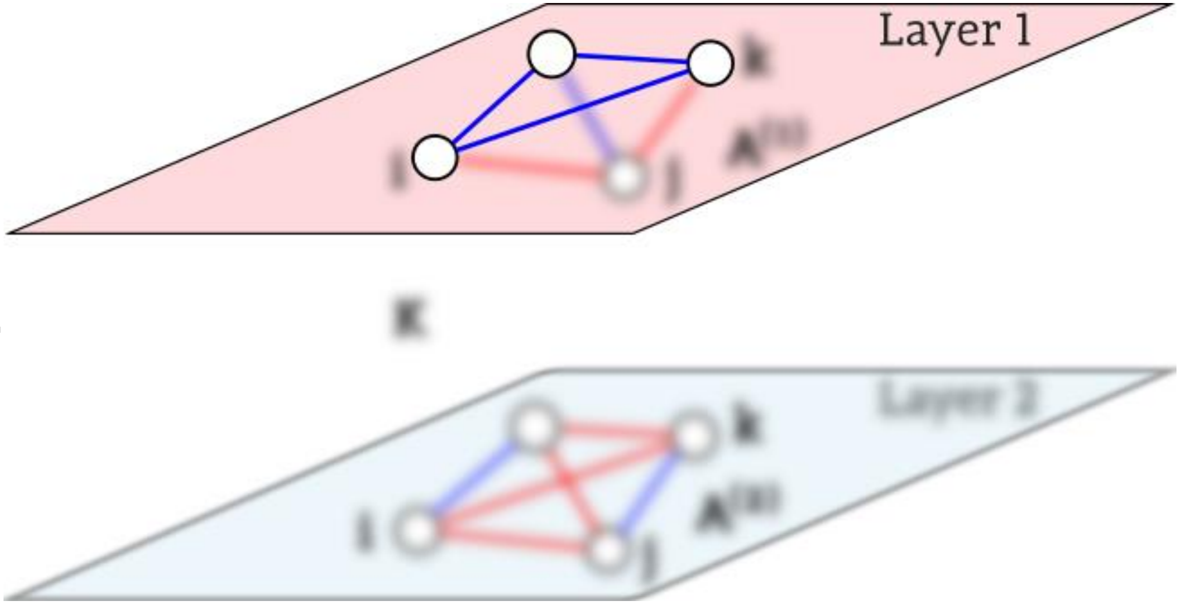

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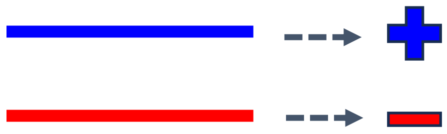

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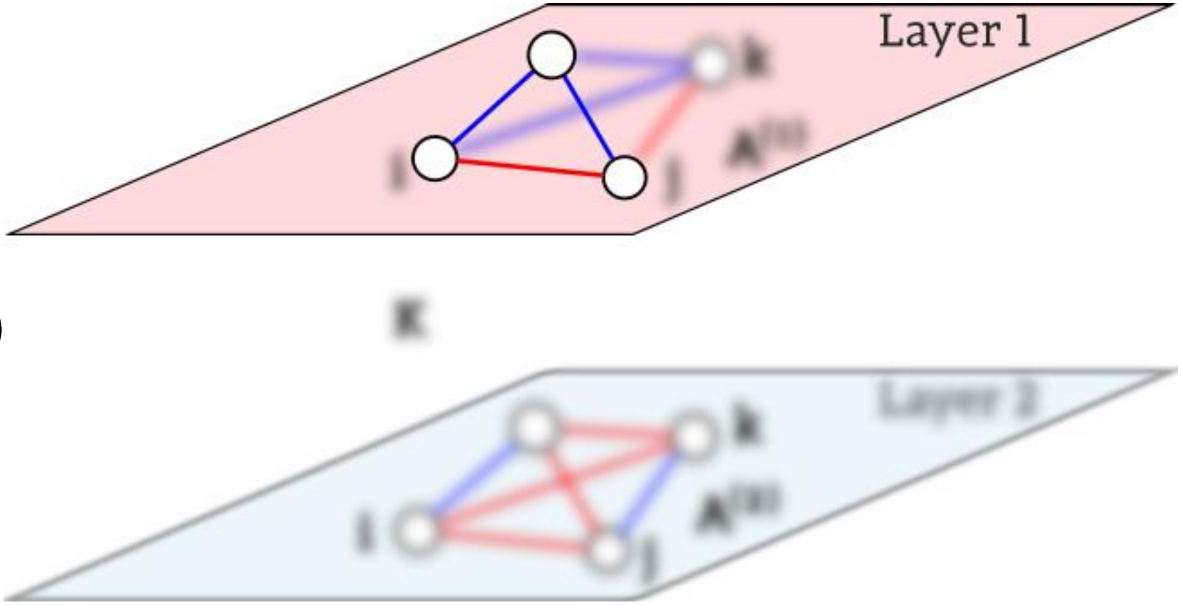

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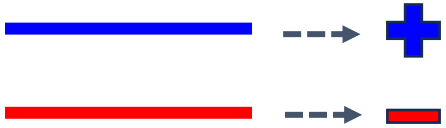

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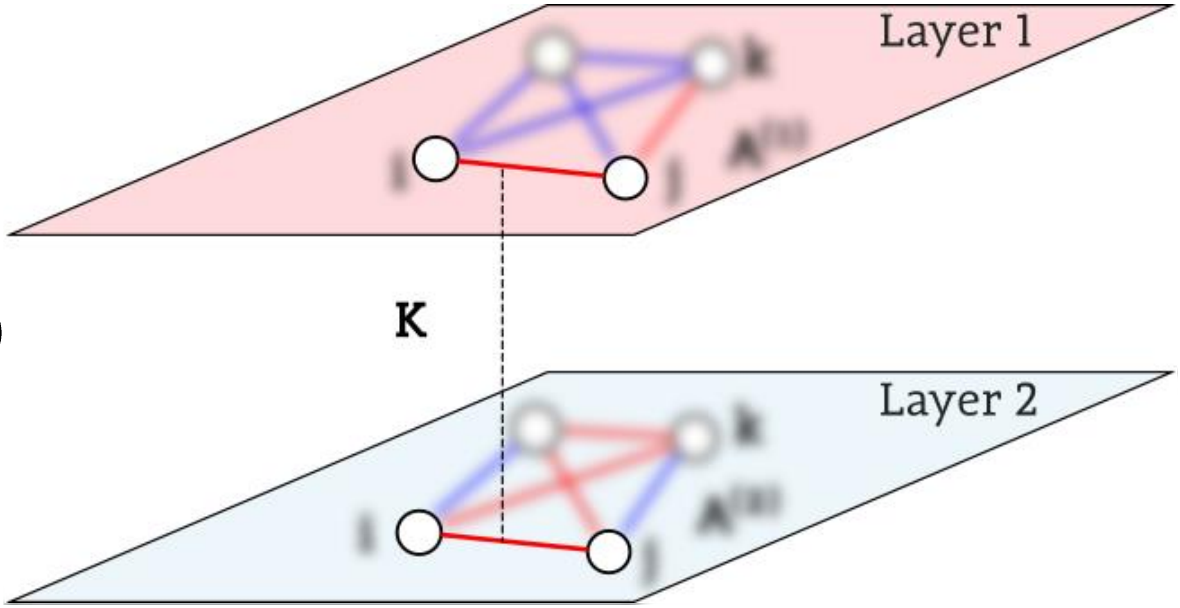

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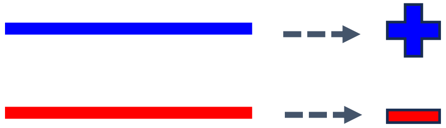

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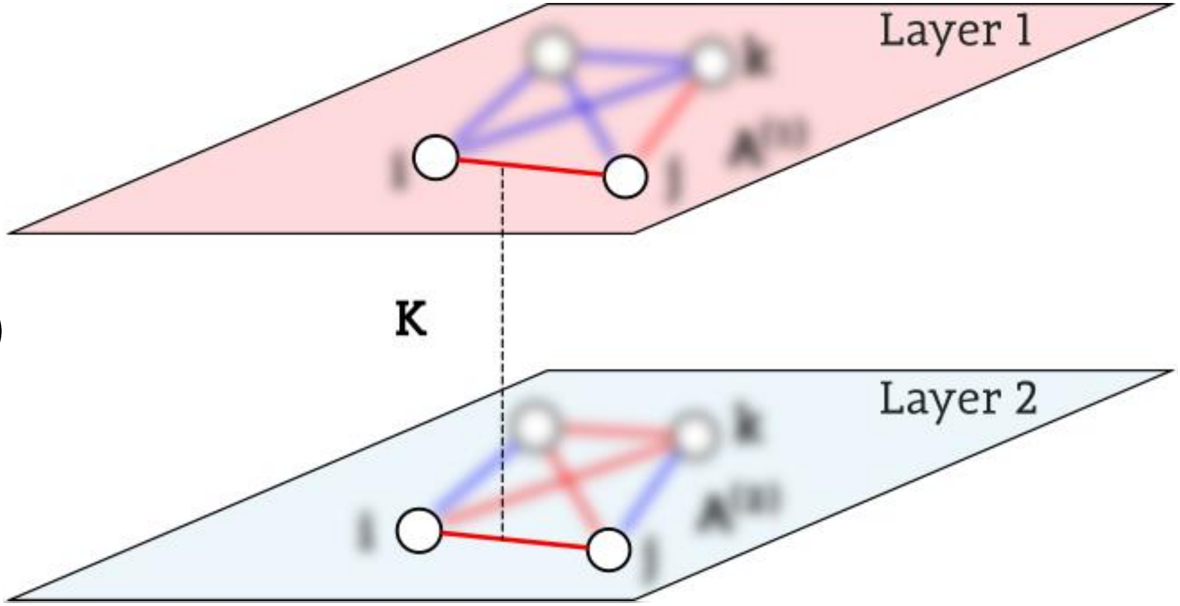
Ising Interactions (between the layer)

Complete graph !



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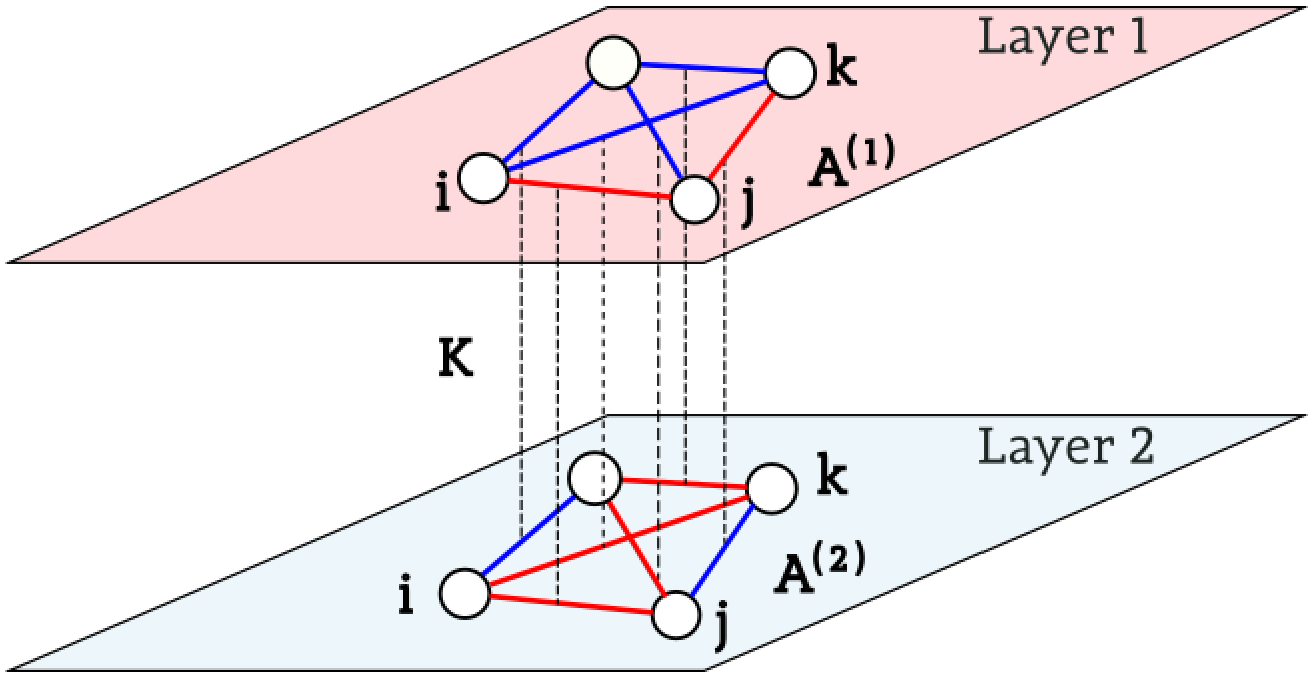
How the connection evolve

What happens to the links when the system evolves?



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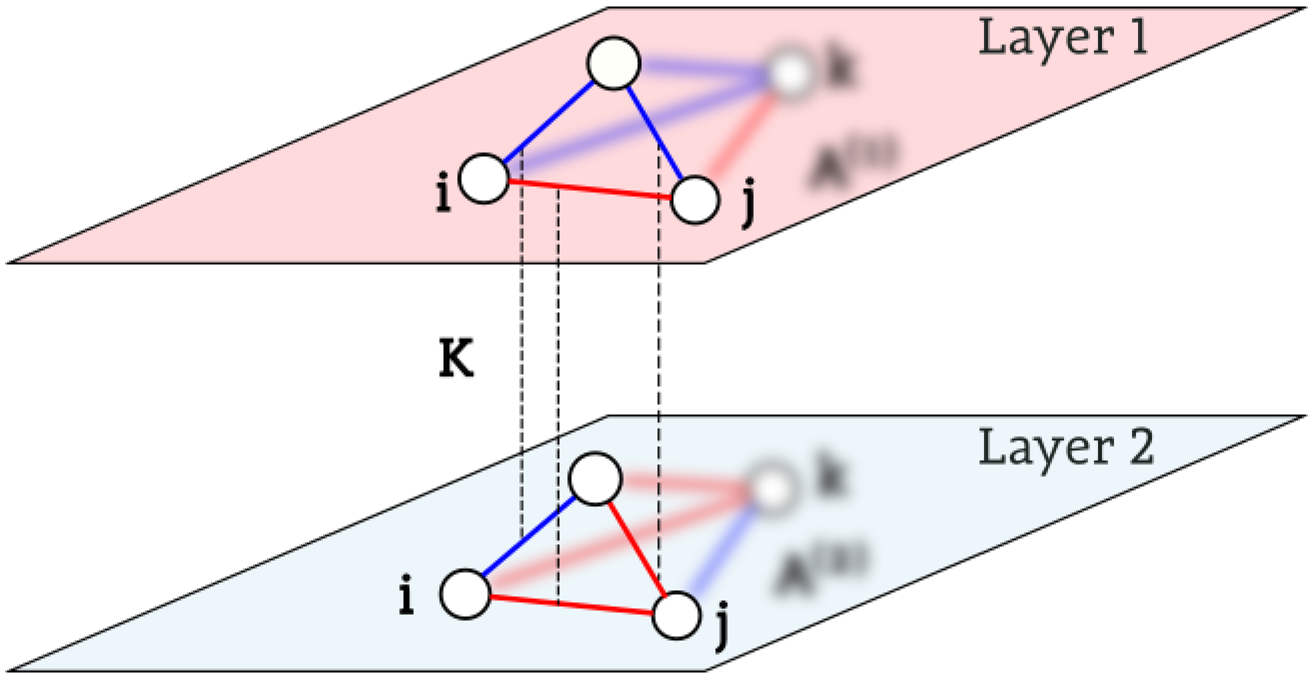
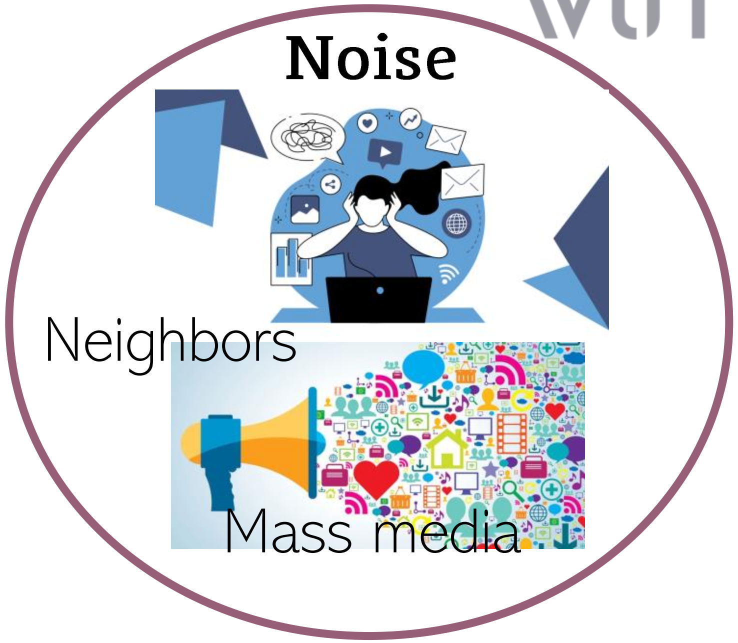
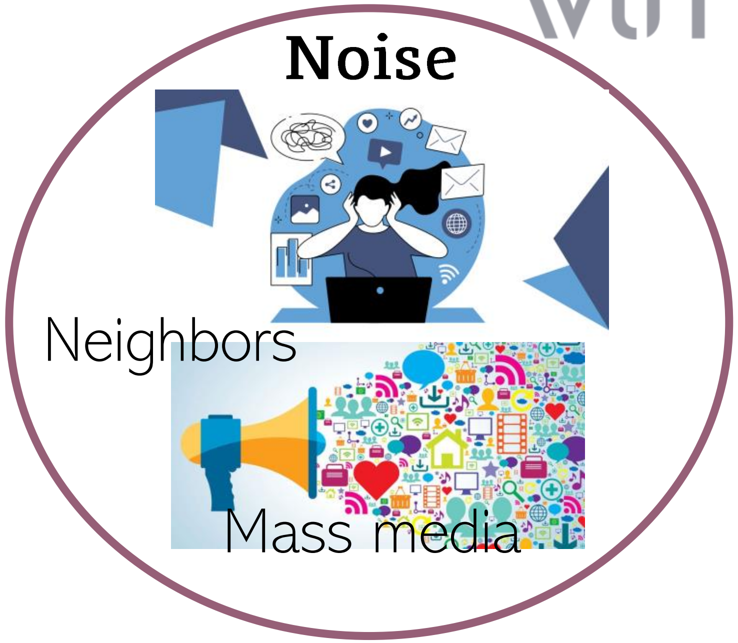


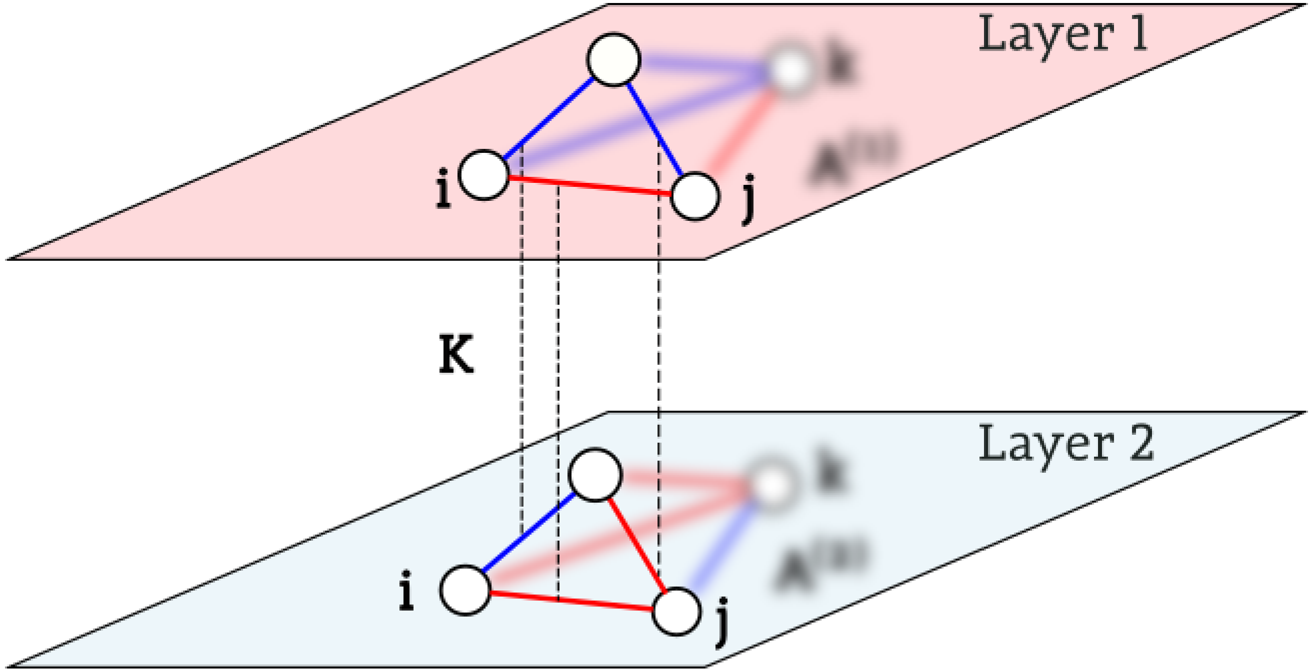


How the connection evolve

What happens to the links when the system evolves?

Temperature





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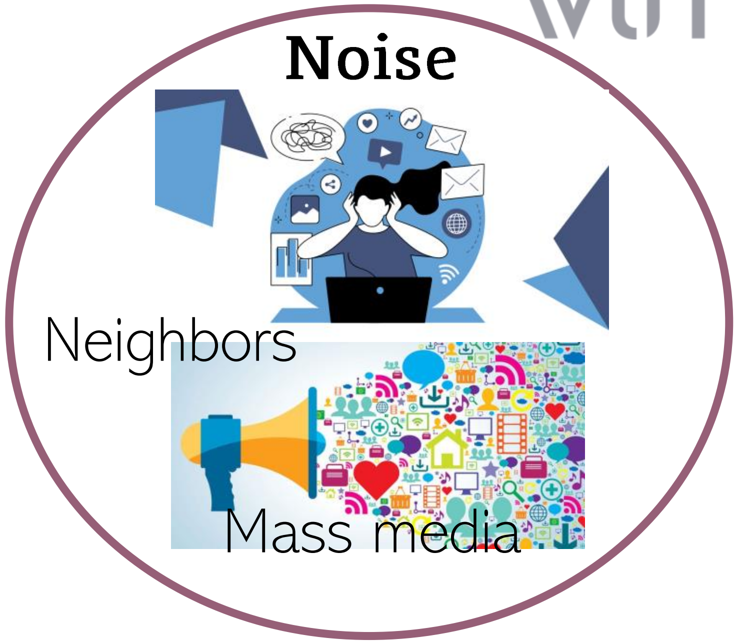

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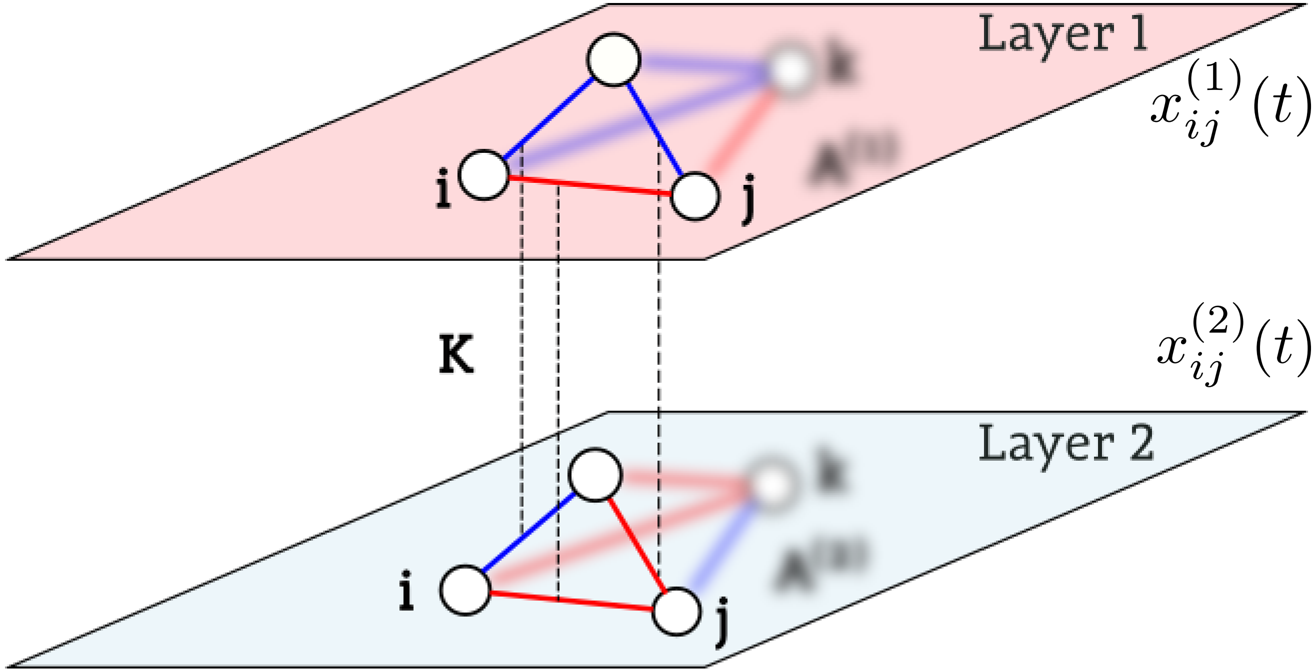


$$x_{ij}^{(1)}(t) = -1 \xrightarrow{T} x_{ij}^{(1)}(t+1) = ?$$

$$x_{ij}^{(2)}(t) = -1 \xrightarrow{T} x_{ij}^{(2)}(t+1) = ?$$

$$p(x_{ij}^{(\alpha)}) = \frac{1}{Z} exp(-H/T)$$

Analytical Approach:

MEAN FIELD APPROXIMATION

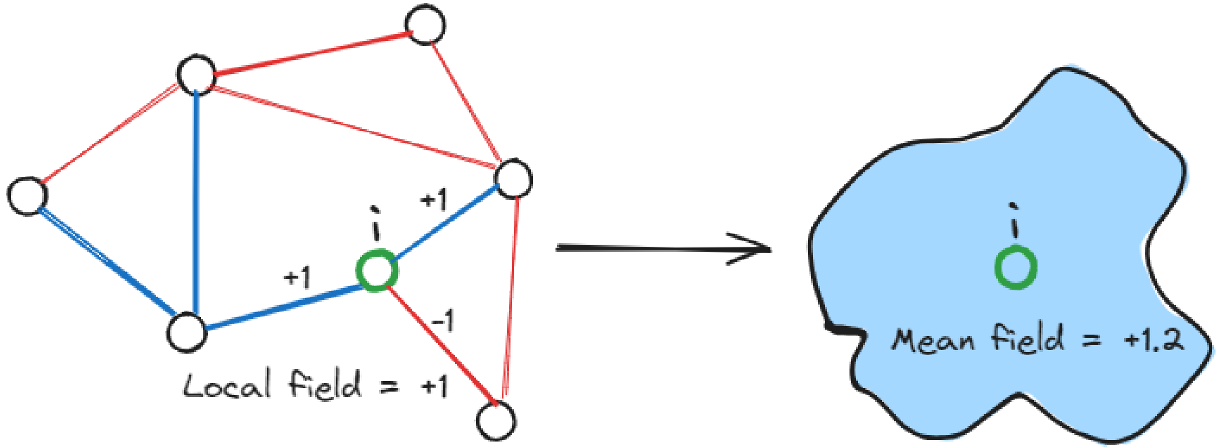
Microstate

Mesostate

Analytical Approach:

- Replacing individual interactions with a single effective interaction representing the average effect of the system.

MEAN FIELD APPROXIMATION



MF Approximation :Microstate

For a duplex network

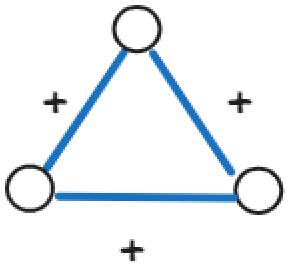
Elementary Subsystems: Pair of coupled links $\vec{x}_{ij} = [x_{ij}^{(1)}, x_{ij}^{(2)}]$

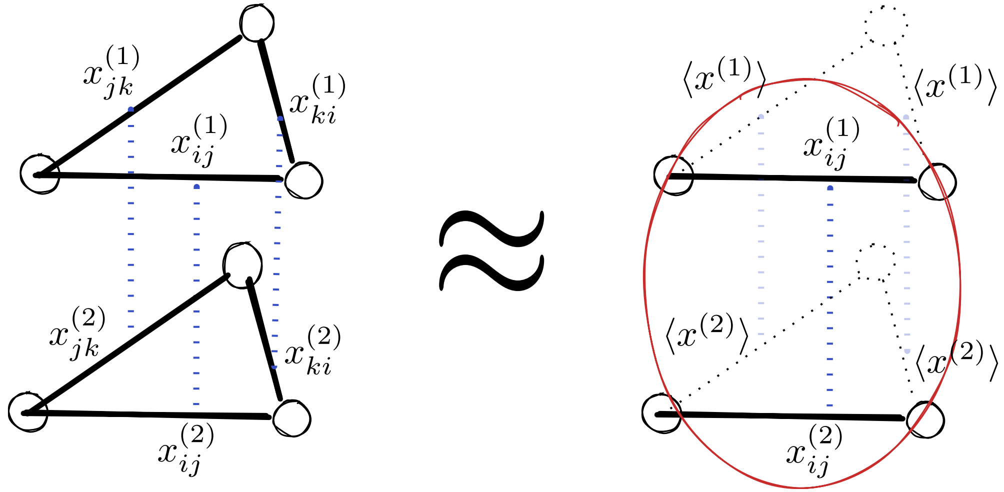
Interactions: These pairs experience interlayer interactions proportional to $[\langle x^{(1)} \rangle^2, \langle x^{(2)} \rangle^2]$

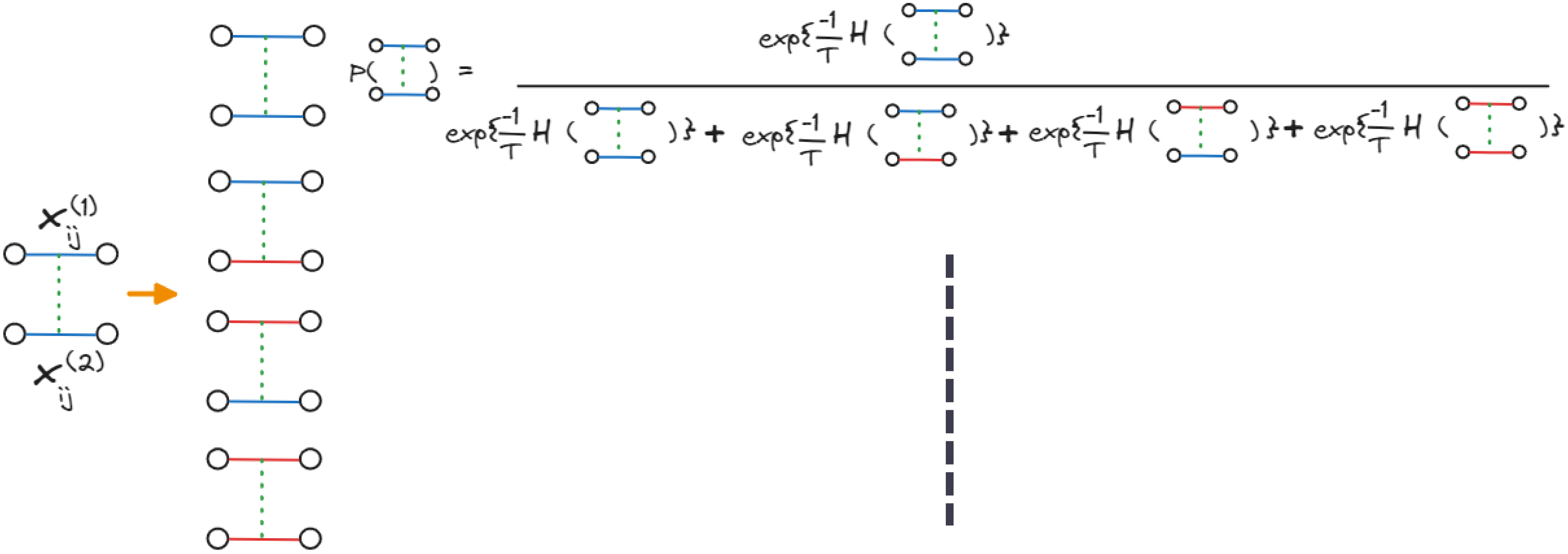
The probabilities of states of this pair are described by the **canonical ensemble** $P(\vec{x}_{ij}) = \frac{\exp(-E(\vec{x_{ij}})/T)}{\sum_{\vec{x}_{mn}} \exp(-E(\vec{x}_{mn})/T)}$

$$E(\vec{x}_{ij}) = -A(N-2)x_{ij}^{(1)}\left(x^{(1)}\right)^2 - A(N-2)x_{ij}^{(2)}\left(x^{(2)}\right)^2 - Kx_{ij}^{(1)}x_{ij}^{(2)}$$

Average polarization $\langle \vec{x}_{ij} \rangle = \sum_{\vec{x_{ij}}} P(\vec{x_{ij}})\vec{x}_{ij}$







$x_{ij}^{(1)}$
$x_{ij}^{(2)}$
P( ) =
exp{-1/T H ( )}

Mean field solution

Self consistency equations:

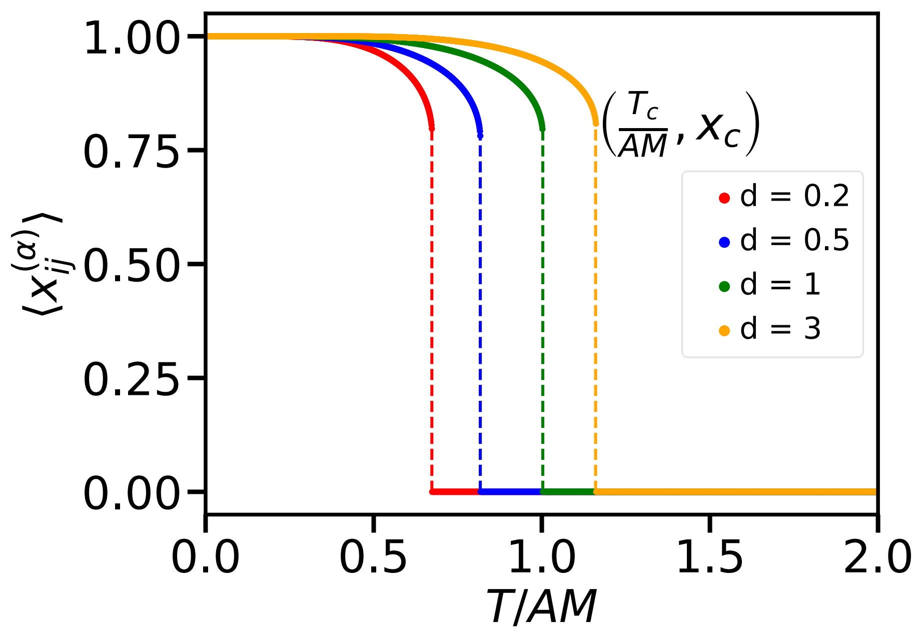
$$x^{(1)} = f_1(x^{(1)}, x^{(2)})$$
$$x^{(2)} = f_2(x^{(1)}, x^{(2)})$$

$$\boxed{f_\alpha(x^{(1)}, x^{(2)}) = \frac{e^{2d}\sinh(a[(x^{(1)})^2+(x^{(2)})^2]) + \sinh(a(-1)^\alpha[(x^{(2)})^2-(x^{(1)})^2])}{e^{2d}\cosh(a[(x^{(1)})^2+(x^{(2)})^2])+\cosh(a[(x^{(1)})^2-(x^{(2)})^2])}}$$

$a = \frac{AM}{T}$ $\quad d = \frac{K}{AM}$ ➡ Re-scaled intralayer and interlayer interaction strength

Observations:

- **Temperature and Polarization:** Increasing temperature T leads to a continuous decrease up to x_c.
- **Critical Point:** At Tc, the mean polarization abruptly jumps to zero, indicating a first-order transition.
- **Unidirectional Transition:** This first-order transition always goes from a polarized to an unpolarized state. Reversed transition does not occur;
- **Dependence on Coupling Strength K:** When increasing the coupling strength (K), the critical temperature (Tc) also increases,



Finding critical temperature

Dynamical equations:
$$x^{(1)}(t+1) = f_1(x^{(1)}(t), x^{(2)}(t))$$
$$x^{(2)}(t+1) = f_2(x^{(1)}(t), x^{(2)}(t))$$

Jacobian to find largest Eigenvalue

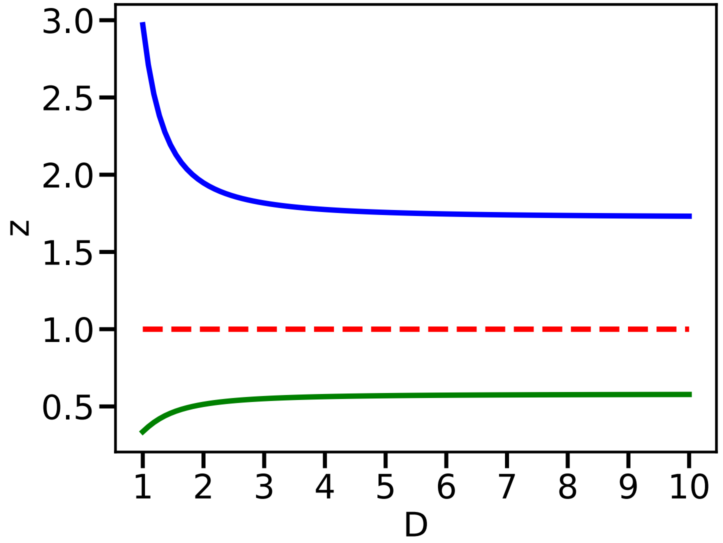
$$x^{(1)} = x^{(2)} \equiv x$$
$$\lambda_+ = 1$$

$$z = e^{ax^2}$$
$$D = e^d$$

Calculate the largest value of *z* for a fixed value of *D*

$$8 \ln z = \frac{(z^4-1)(z^4D^2+2z^2+D^2)}{z^2(z^4+2z^2D^2+1)}$$

$$\boxed{\frac{T_c}{AM} = \frac{x_c^2}{\ln z} = \left(\frac{1}{\ln z}\right)\left(\frac{D^2(z^4-1)}{D^2(z^4+1)+2z^2}\right)^2}$$



MF Approximation

Microstate

- Set of links between i and j.
- L number of self consistency equation
- Lower order (duplex, triplex network)

MF Approximation

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Mesostate

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- Number of positive links L^{+} among all L links in the set.
- One MF equation.
- Higher order multiplex network

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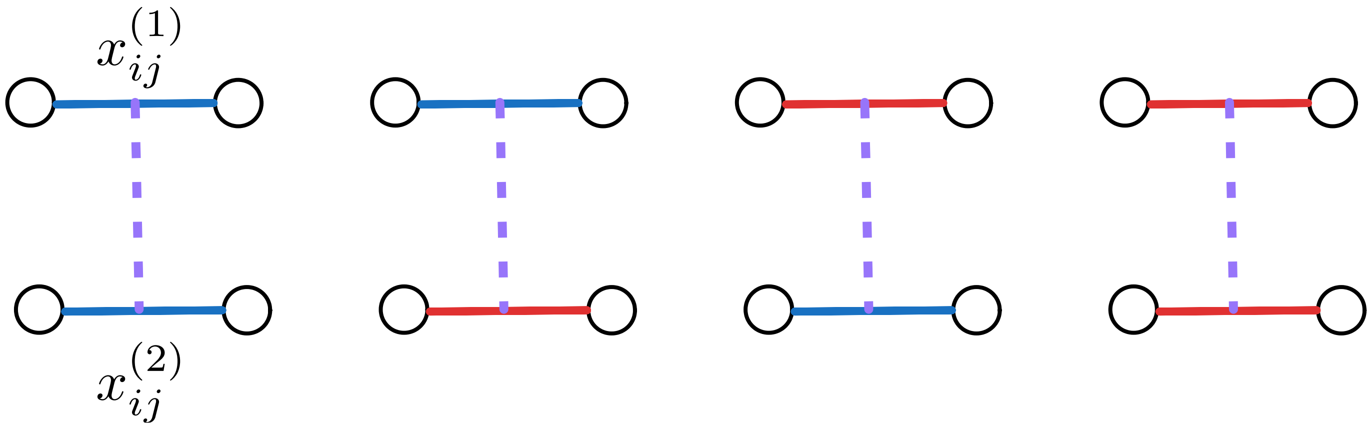

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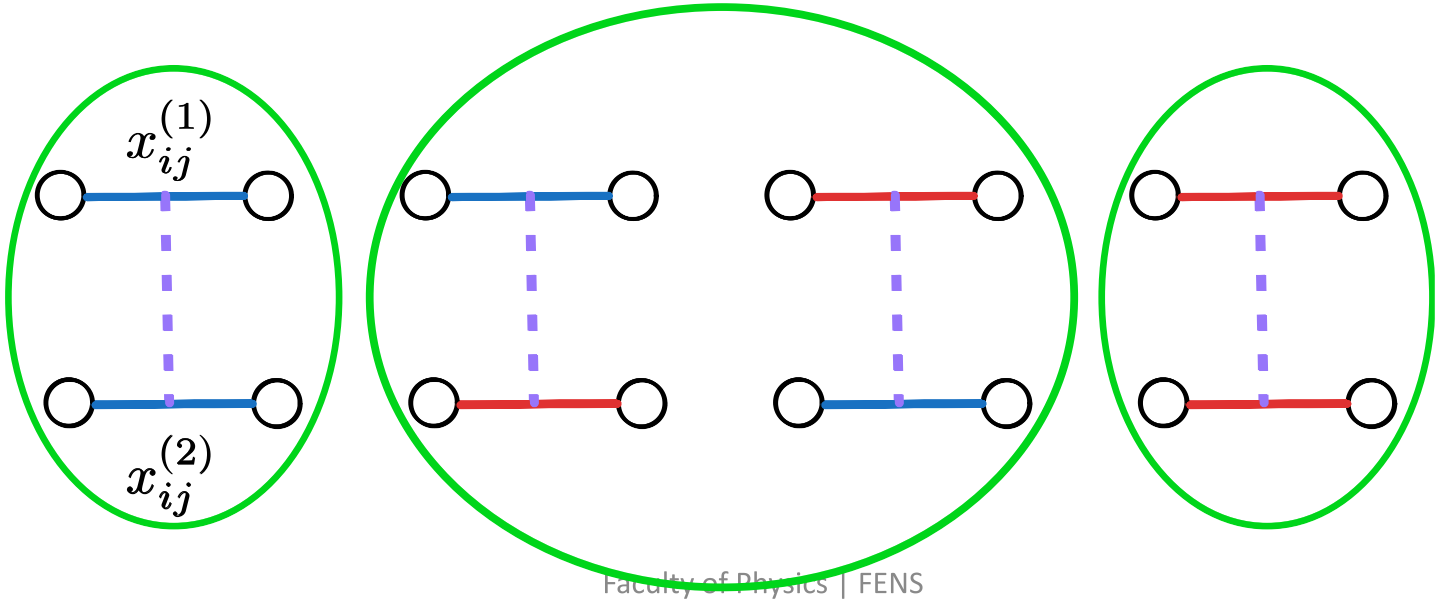


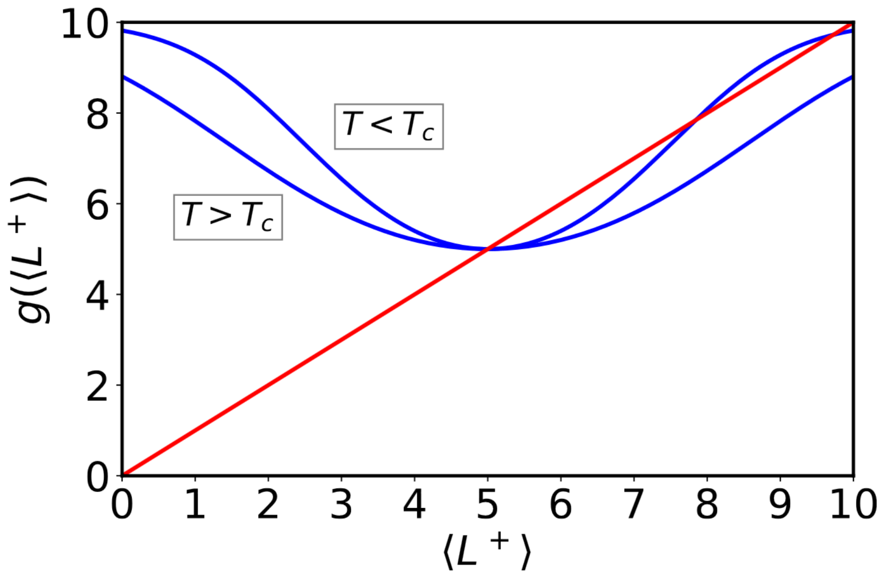
MF Approximation : mesostate

Mean link polarization $\boxed{\langle x \rangle = \left\langle \frac{L^+ - L^-}{L} \right\rangle = \frac{2\langle L^+ \rangle}{L} - 1}$

$$\mathcal{M} = \binom{L}{L^+}$$

Self-consistent equation

$$\boxed{\langle L^+ \rangle = g(\langle L^+ \rangle) = \frac{\sum_{L^+=0}^{L} L^+ \binom{L}{L^+} e^{-E(L^+)/T}}{\sum_{L^+=0}^{L} \binom{L}{L^+} e^{-E(L^+)/T}}}$$

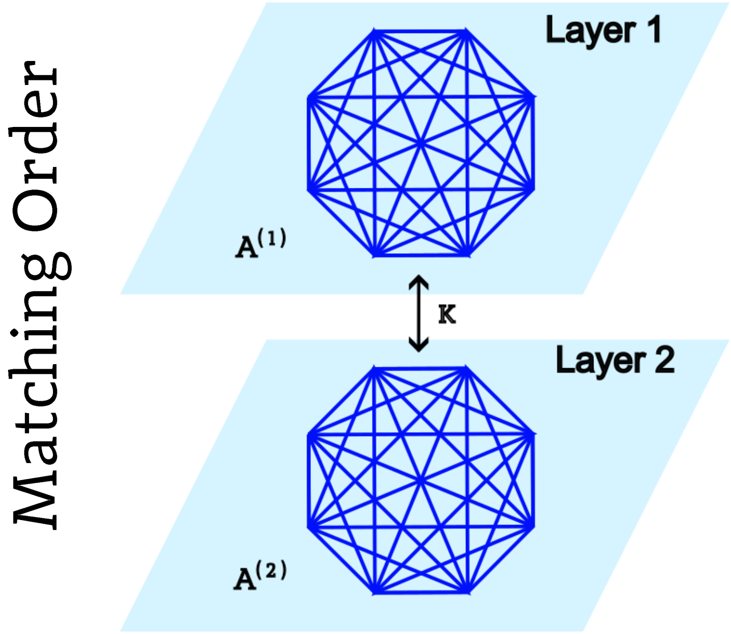


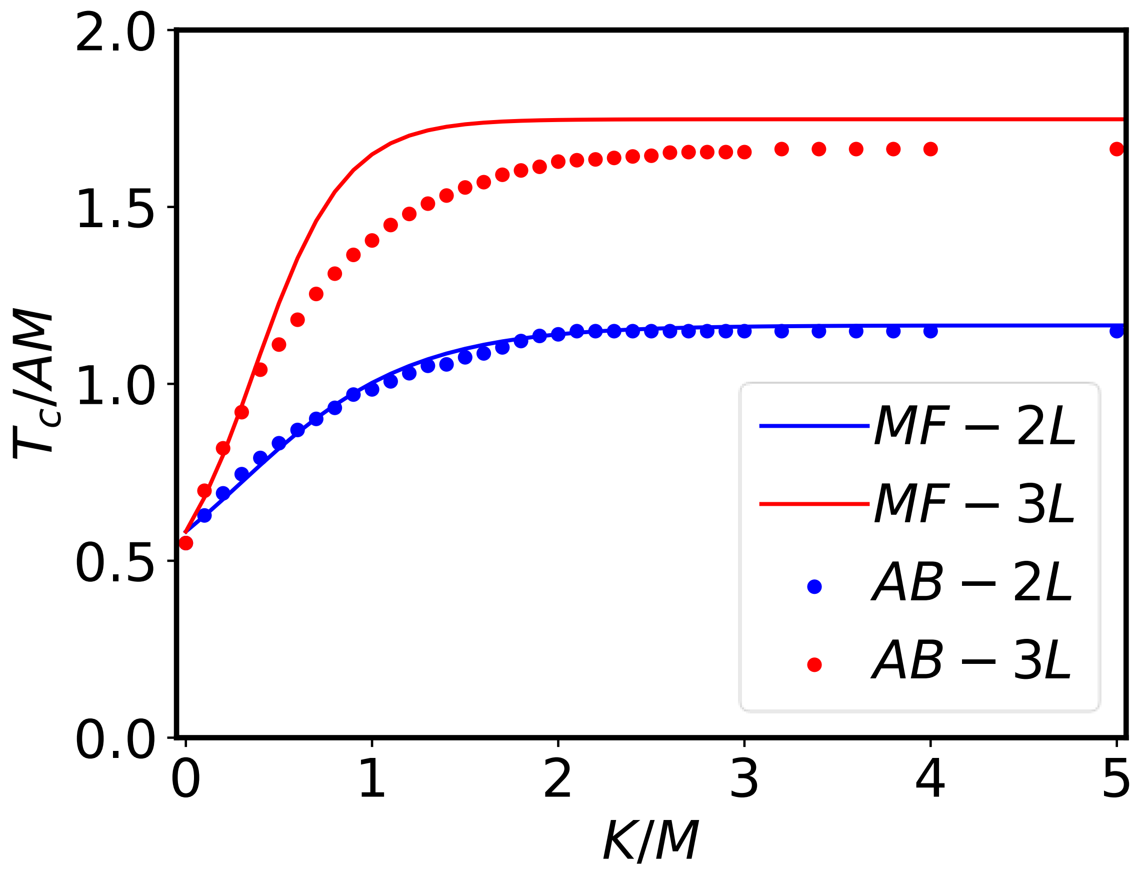
$\langle L_+ \rangle = L$ is a paradise state, and $\langle L_+ \rangle = L/2$ is a disordered state

Saturation of critical temperature

For large coupling strength K-

$$\boxed{T_c\big|_{\alpha=L, K\to+\infty} \approx L \cdot T_c\big|_{\alpha=1}}$$

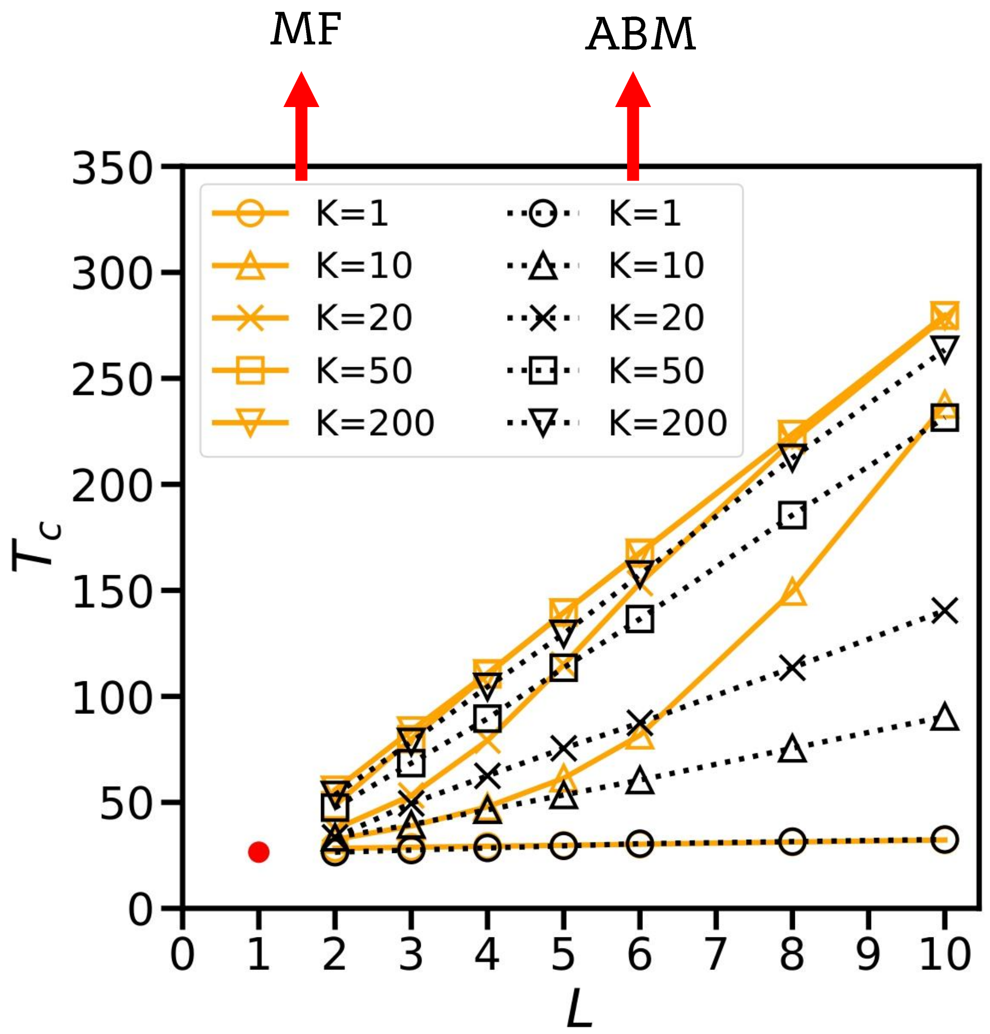




Impact of number of Layers on the Critical temperature

The critical temperature Tc increases with the number of layers L when coupling K is positive

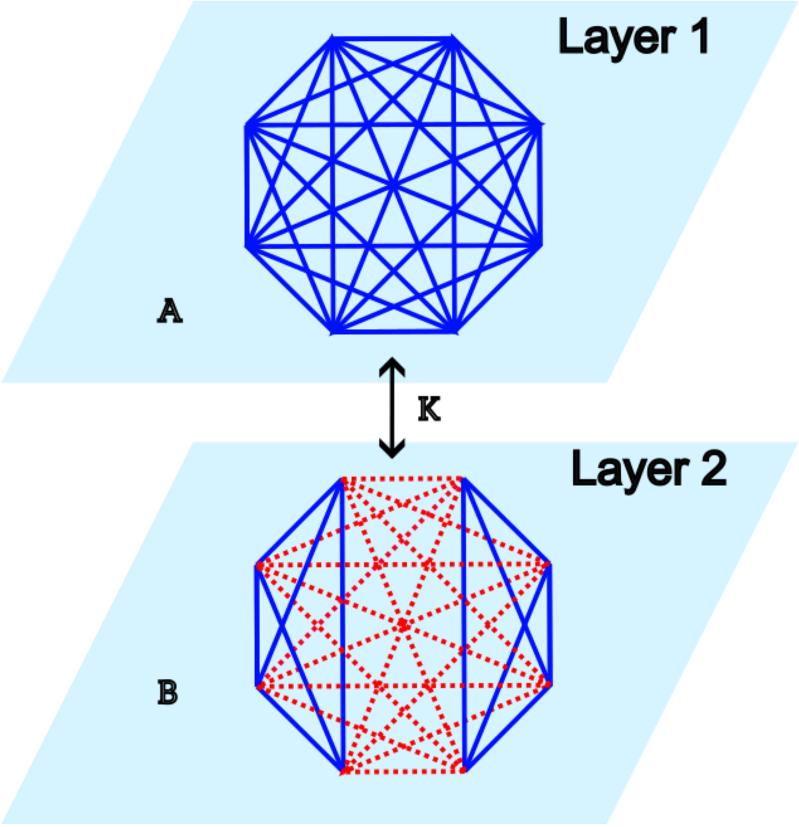
The mean-field approach, while qualitatively correct, is not capable of accurately predicting the behavior of a system with numerous weakly interacting layers.



Critical behavior of the system with layers in different states

What happens when two layers start differently?

Mismatched Order



N

Paradise → $\langle x_{ij}^{(2)} \rangle = 1$

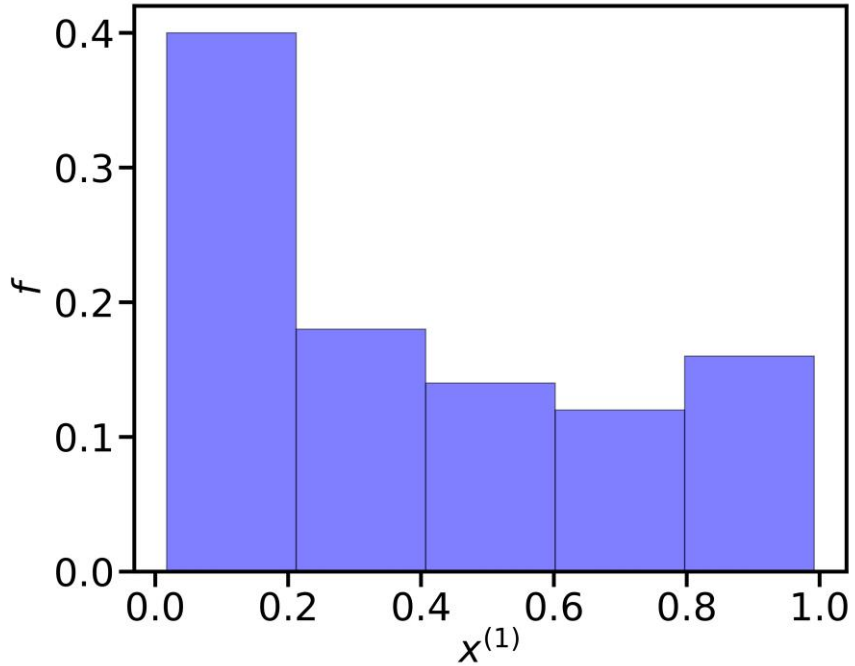
- Heider (Intralayer) Energy in ground state
- Ising (Interlayer) Energy in Excited state

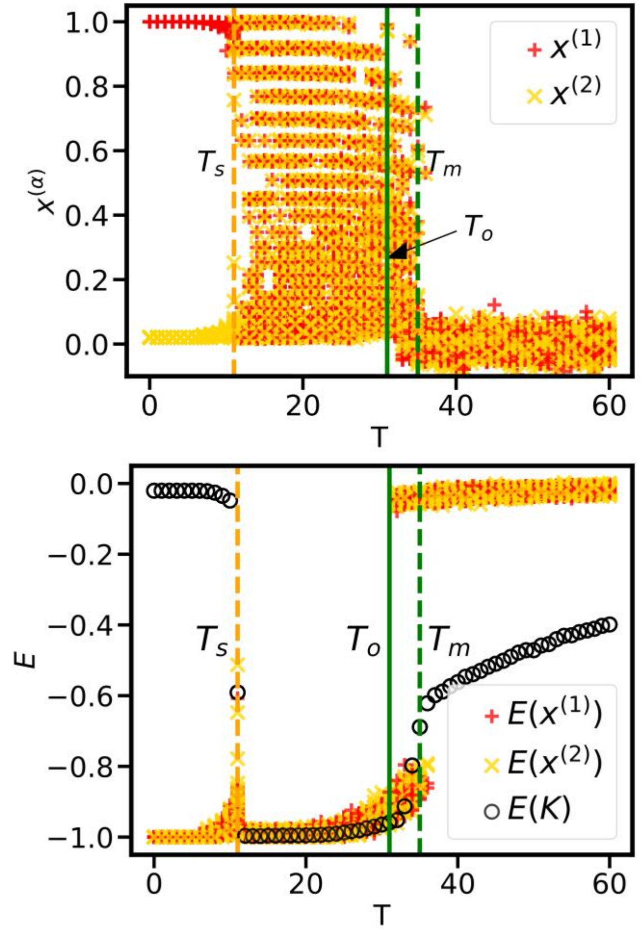
2-Clique → $\langle x_{ij}^{(2)} \rangle = 0$

$N = 2m$

Critical behavior of the system with layers in different states

- Different layers may exist in different states
- Possibilities include one layer ordered and the other disordered, or variations in order within both layers.





Synchronization between the layers!

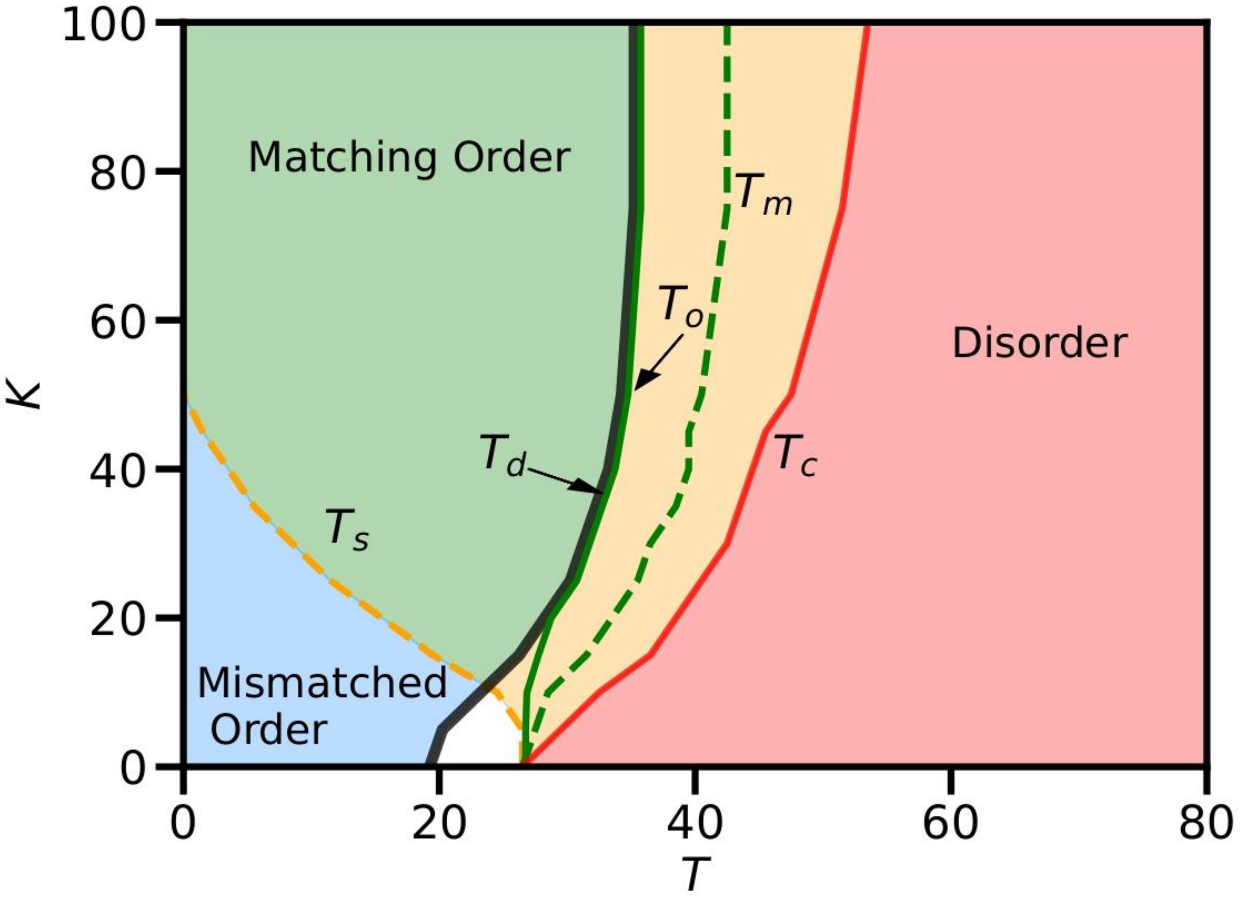
Critical behavior of the system with layers in different states

Temperature T_s:
Critical temperature where different states of layers synchronize.

T_m: The limit of retaining order after synchronization

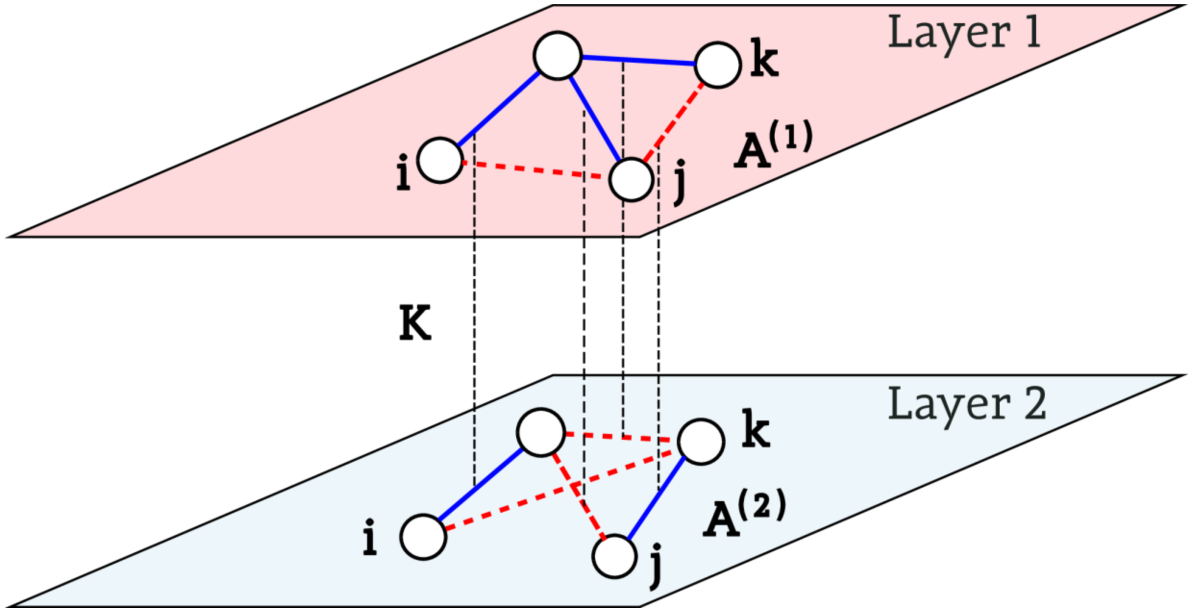
T_o: The limit of spontaneous order

T_c: Critical temperature for the existence of ordered states



Not everyone knows everyone !

Missing links → $x_{ij} = 0$ (not a true link state)



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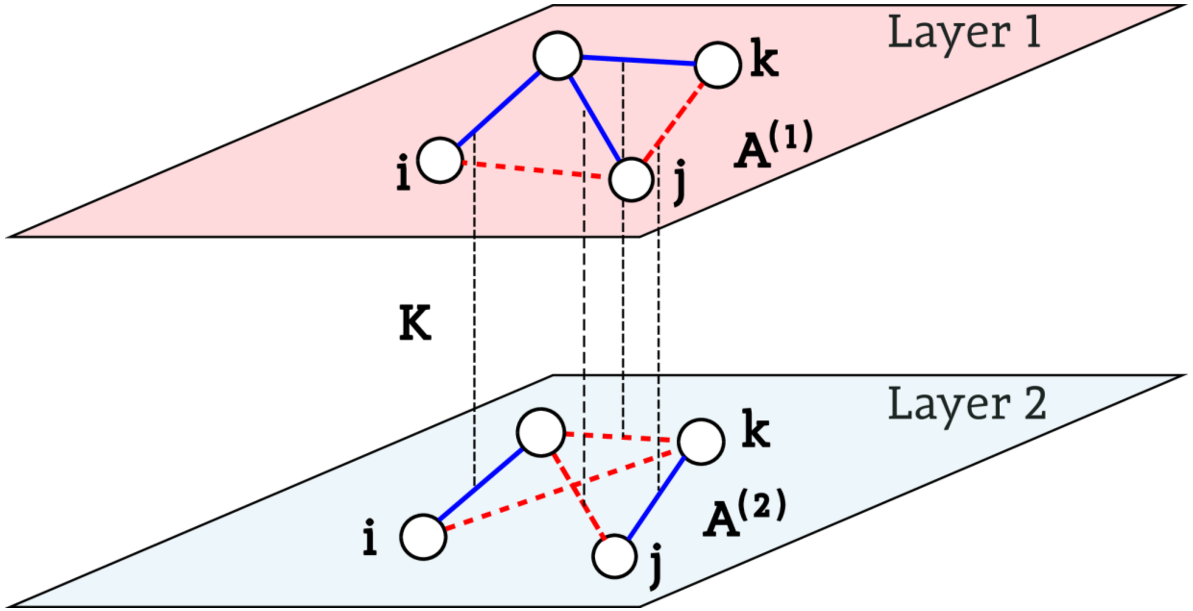
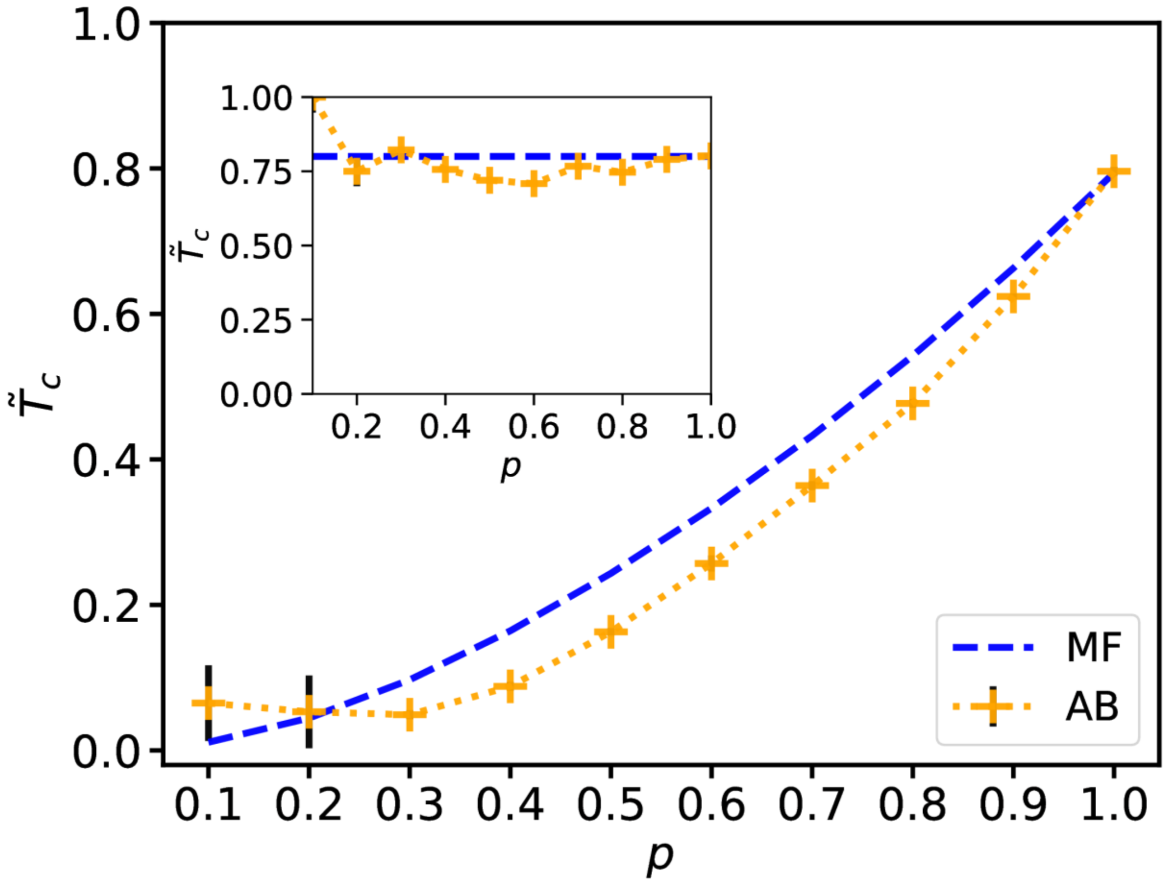


Erdös-Rényi graph !

Heider interaction strength scales with p^2, while interlayer interaction strength scales with p.

Critical temperature



$$T_c = Ap^2 M \left(\frac{1}{\ln z}\right) \left(\frac{D^2(z^4-1)}{D^2(z^4+1)+2z^2}\right)^2$$

But T_c not increases like p^2 !

Important results

Social science Lens

Transition from a state of stability to unrest

Tendency of relationships in one layer to align with those in the other.

Influence of balanced or imbalanced relationships becomes increasingly pronounced.

<u>Complete graph</u>

Transition

Duplex network with contrasting relationship layers

<u>Random graph</u>

Interaction

Physics Lens

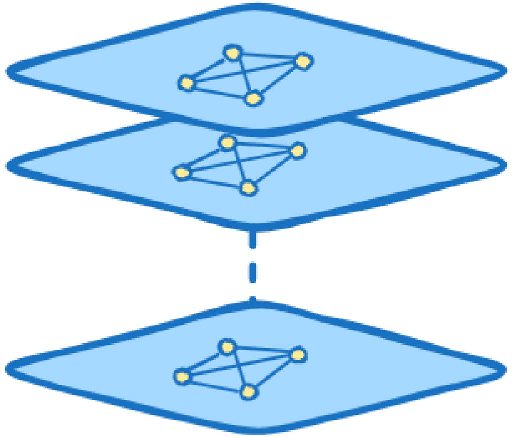
Temperature rise causes a multiplex network's shift from paradise to disorder, surpassing single-layer critical temperatures.

Synchronization of interlayer relations

Heider interaction scales like p^2
Interlayer interaction scales like p.

Main Message

Building several layers of social interactions make the social structures more stable



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[1] Mohandas, K., Suchecki, K. and Hołyst, J.A., 2024. Physical Review E, 109(4), p.044306.
[2] P. J. Górski, K. Kułakowski, P. Gawro´ nski, and J. A. Hołyst, Sci. Rep. 7, 16047 (2017).

Thank you