On the design of public debate in social networks

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  - social influence on individuals decay over time ($\rightarrow$ intensity of social debate, or speed at which opinion cristalyze) (similar to DeMarzo, Vayanos and Zwiebel (2003))

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    2. the diameter of opinions decreases with the intensity of social debate
  - trade-off between fostering the convergence of opinions and the risk of polarization from the view-point of the social planner

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$$G_{ij}(t) := \begin{cases} \dfrac{1}{\deg(i, \mathcal{G}(t))} & \text{if } \{i, j\} \in \mathcal{G}(t) \\ 0 & \text{otherwise} \end{cases}$$

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- $N_{\mathcal{G}}(i)$: neighborhood of i in the network $\mathcal{G}$

Dynamics of the network and of opinions

1. Opinion dynamics:

$$x(t+1) = [\lambda^t G(t) + (1-\lambda^t)I]x(t)$$

with $\lambda \in [0,1]$ (intensity of social debate; (inverse of) speed at which opinions crystalize) and I the identity matrix.

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$$\begin{aligned}\mathcal{G}(t+1) = \mathcal{G}(t) \cup \{\{i,j\} \text{ s.t. } |x_i(t+1) - x_j(t+1)| < \tau\} \\ \setminus \{\{i,j\} \text{ s.t. } |x_i(t+1) - x_j(t+1)| \geqslant \sigma\}\end{aligned}$$

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2. $\sigma = \tau$, $\lambda = 1$: Hegselmann-Krause model

Compatible opinion networks

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  - These intervals are ordered and never nested: For all i, j

$$x_i \leqslant x_j \Rightarrow (x_{n_-(i)} \leqslant x_{n_-(j)}) \text{ and } (x_{n_+(i)} \leqslant x_{n_+(j)})$$

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$$\{i, j\} \in \mathcal{G} \Rightarrow |x_i - x_j| < \tau \quad \text{and} \quad \{i, j\} \notin \mathcal{G} \Rightarrow |x_i - x_j| \geqslant \sigma.$$

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- (σ, τ)-compatible implies compatible.

Compatible opinion networks

Lemma

Let $(x(t), \mathcal{G}(t))$ be a compatible network-opinion pair. Then for any network formation process, one has:

1. *If $x_i(t) = x_j(t)$ then $x_i(t') = x_j(t')$ for all $t' \geqslant t$.*
2. *$x_i(t) < x_j(t)$ implies $x_i(t') \leqslant x_j(t')$ for all $t' \geqslant t$, with strict inequality if $\lambda \neq 1$.*
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In the following, we assume that agents are initially numbered so that

$$x_1(0) < x_2(0) < \cdots < x_n(0)$$

Due to the above Lemma, they will always remain ordered in this way.

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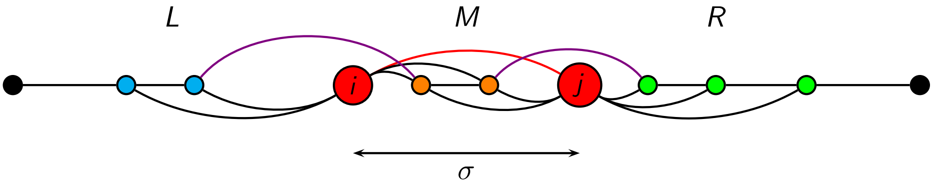

Link fragility

Definition

Using the above notation, the fragility of a (maximal) link $\{i, j\}$ in $\mathcal{G}$ is defined by:

$$\phi_{i,j}(\mathcal{G}) = \max_{m \in \{0,\dots,M\}} \frac{(L+M+2)(2R-r_m)+(R+M+2)(L-\ell_{m+1})+(L-R)(M-m+1)}{(L+M+2)(R+M+2)} \quad (1)$$

letting $r_0 = 0$ and $\ell_{M+1} = 0$.

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Proposition

Let σ,τ,λ be given and $\{i,j\}$ be a maximal link in the network $\mathcal{G}(t)$. Then one has $\{i,j\} \in \mathcal{G}(t+1)$ for any $x(t)$ that is (σ,τ)-compatible with $\mathcal{G}(t)$ if and only if $\phi_{i,j}(\mathcal{G}(t)) \leqslant 1$.

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Proposition

If the initial network $\mathcal{G}(0)$ has no fragile link, then for all $\lambda \in [0, 1]$, one has $\mathcal{G}(t) = \mathcal{G}(0)$ for all $t \in \mathbb{N}$, i.e., the network is stable for the dynamics independently of the choice of a compatible $x(0)$.

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Answer: Supposing that the difference of opinions on a link is bounded above by $\rho < \sigma$ and that $\frac{\sigma}{\rho} \geq \overline{\phi}_{\mathcal{G}(0)}$, the network is stable provided λ is smaller than a constant depending on $\overline{\phi}_{\mathcal{G}(0)}$ and σ/ρ.

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- Letting $\overline{\psi}_{\mathcal{G}(0)}$ be the maximum over $\phi_{i,i+1}$ for $\mathcal{G}(0)$, it can be proven that if the initial network $\mathcal{G}(0)$ is connected and satisfies $\overline{\psi}_{\mathcal{G}(0)} \leq 1$, then $\mathcal{G}(t)$ remains connnected for all $t \in \mathbb{N}$ and for all $\lambda \in [0, 1]$, independently of the choice of a compatible $x(0)$.

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- Similarly to network stability, in the case that $\overline{\psi}_{\mathcal{G}(0)} > 1$, and supposing that $x_{i+1}(0) - x_i(0) \leq \rho < \sigma$ for $i = 1, \ldots, N-1$, and $\frac{\sigma}{\rho} \geq \overline{\psi}$, the network $\mathcal{G}(t)$ remains connected for all $t \in \mathbb{N}$ if λ is smaller than a constant depending on $\overline{\psi}_{\mathcal{G}(0)}$ and σ/ρ.

Simulations

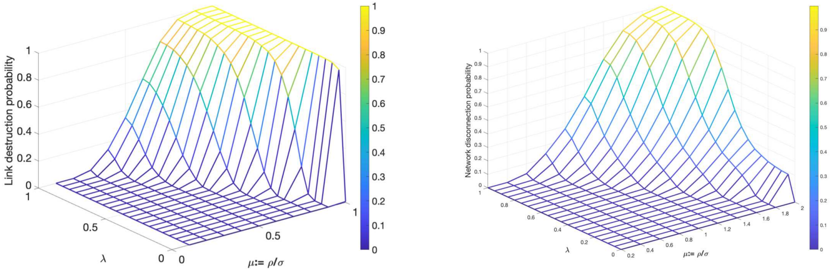

Figure: Empirical probability (computed over 50,000 graphs with various initial opinions and edge-connectedness) of the destruction of at least one link (left panel) and of the disconnection/polarization of the network (right panel) for various levels of λ and $\mu = \rho/\sigma$.

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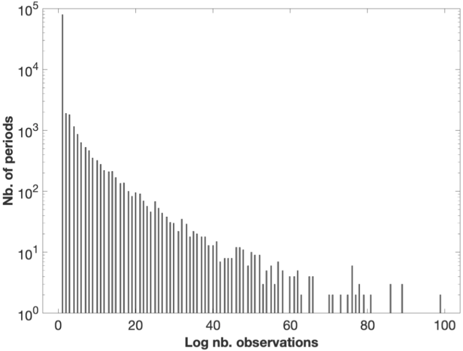

Figure: Histogram of the time to disconnection/polarization for networks in the sample that actually disconnects.

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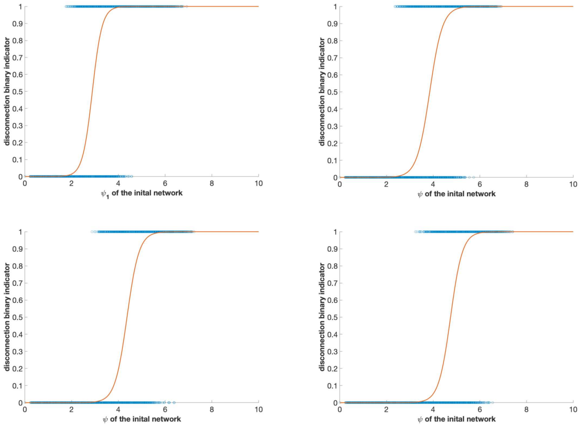

Figure: Scatter plot of a binary indicator of disconnection (equal to 1 if the network disconnects/polarizes in the course of the simulation and to 0 otherwise) as a function of the polarizability of the initial network $\overline{\psi}$. Upper left to lower right: edge-connectdness = 1, 2, 3, 4 resp. Logistic regression curve is displayed in red.

Convergence and asymptotic opinions

Proposition

Assume that the network $\mathcal{G}$ is stable and connected, with initial opinion vector $x(0)$, and denote by w the normalized left eigenvector of G associated to eigenvalue 1. Then the opinion vector $x(t)$ can be asymptotically approximated by

$$x(\infty) \approx \Big[\frac{\varphi(\lambda)}{\log(\lambda)}\big(\log(1-\lambda)+\Psi_\lambda(1)\big)G^2+\varphi(\lambda)G$$

$$+\Big(1-\frac{\varphi(\lambda)}{\log(\lambda)}\big(\log(1-\lambda)+\log(\lambda)+\Psi_\lambda(1)\big)\Big)\mathbf{1}w^T\Big]x(0)$$

$$\approx \Big[(1-\varphi(\lambda))\mathbf{1}w^T+\varphi(\lambda)G\Big]x(0).$$

red: consensus part   blue: strong diversity part

$\lim_{\lambda\to 1}\varphi(\lambda)=0$, $\lim_{\lambda\to 0}\varphi(\lambda)=1$

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- It can be shown that the diameter is decreasing with λ:

$$\lambda > \mu \Rightarrow \delta(x_\lambda(t)) \leq \delta(x_\mu(t))$$

(some restrictions are needed for $<$, e.g., G connected and $x(0)$ s.t. $x_1(0) < x_2(0) \cdots < x_n(0)$)

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- The result on convergence shows that, even if there is no polarization, there is no consensus in the society (contrarily to most models)(*strong diversity* vs. *weak diversity*).

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- If λ increases, the diameter shrinks (and so $\mathcal{R}$ is *a fortiori* winning), but at some point, the society disconnects: the voters in favor of $\mathcal{L}$ are no more under the influence of $\mathcal{R}$: the minority candidate has incentives to exacerbate the public debate so as to separate his/her electorate from the majority.

Campaign strategy and polarization

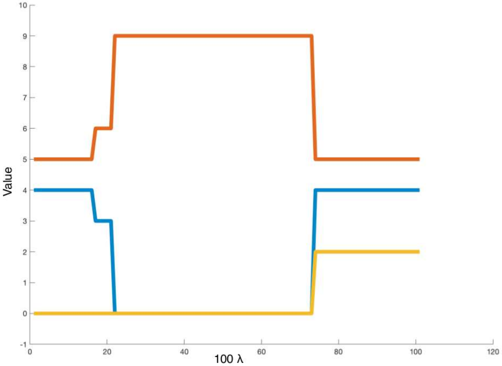

Figure: Values of $u_{\mathcal{L}}(x_\lambda(\infty))$ (blue), of $u_{\mathcal{R}}(x_\lambda(\infty))$ (red) and of the number of connected components (yellow) in G_λ for λ varying between 0 and 1.

Thank you for your attention!

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- We suppose that the network has reached or is initialized in a state $(x(0), \mathcal{G}(0))$ such that no link is fragile, and link creation is discarded.
- We suppose $\lambda \in]0, 1[$ and want to know the evolution of $x(t)$.
- We have:

$$\begin{aligned} x(t+1) &= [\lambda^t G + (1-\lambda^t)I]x(t) \\ &= M(t)x(t) = M(t)M(t-1)\cdots M(1)x(1) = H(t)x(1) \end{aligned}$$

with $x(1) = Gx(0)$, and

$$H(t) = \prod_{k=1}^{t} (\lambda^k G + (1-\lambda^k)I).$$

Hence, $H(t)$ is a polynomial in G of degree t, i.e., of the form:

$$H(t) = h_{t,t}G^t + h_{t-1,t}G^{t-1} + \cdots + h_{1,t}G + h_{0,t}I.$$

Proposition

The coefficients of the polynomial $H(t)$ in G are given by

$$h_{t,t} = \lambda^{t(t+1)/2}$$

$$h_{0,t} = \prod_{i=1}^{t}(1-\lambda^i)$$

$$h_{i,t} = \lambda^{t(t+1)/2} \sum_{1\leq j_1<j_2<\cdots<j_i\leq t} \prod_{\substack{\ell=1 \\ \ell\neq j_1,\ldots,j_i}}^{t} \frac{1-\lambda^\ell}{\lambda^\ell} \quad (0 < i \leq t)$$

with the convention that $\prod_\emptyset = 1$. The asymptotic behavior of the coefficients is given by $\lim_{t\to\infty} h_{i,t} > 0$ if $i \in \mathbb{N}$, while $\lim_{t\to\infty} h_{i(t),t} = 0$ if $i(t) \to \infty$. In particular, we have

$$\lim_{t\to\infty} h_{0,t} = \varphi(\lambda) \qquad (\varphi: \text{ Euler function; } \Psi_\lambda: \ \lambda\text{-Digamma function})$$

$$\lim_{t\to\infty} h_{1,t} = \varphi(\lambda)\Big(1 + \frac{\log(1-\lambda)-\log\lambda}{\log\lambda} + \frac{1}{\log\lambda}\Psi_\lambda(1)\Big).$$