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GRUPO DE SISTEMAS COMPLEJOS



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# Polarization Metrics and Opinions Inference in Multipolar Systems

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# Outline

- Model of Opinion Inference on Networks
  - Bipolar systems
  - Multipolar systems
- Polarization Metrics
  - Bipolar Case: The Polarization Index
  - Multipolar Cases. Covariance matrix
- Applications. Empirical Studies:
  1. Bipolar systems:
    - Second Round 2017 Chilean Presidential elections, 2017
    - Catalanian Independence issue, 2017
    - Venezuela conversation about Hugo Chavez, 2015
  2. Multipolar systems:
    - Spanish elections (2015). Tetrapolar case.
    - Spanish elections (2019). Pentapolar case.

# Model to Estimate Opinions

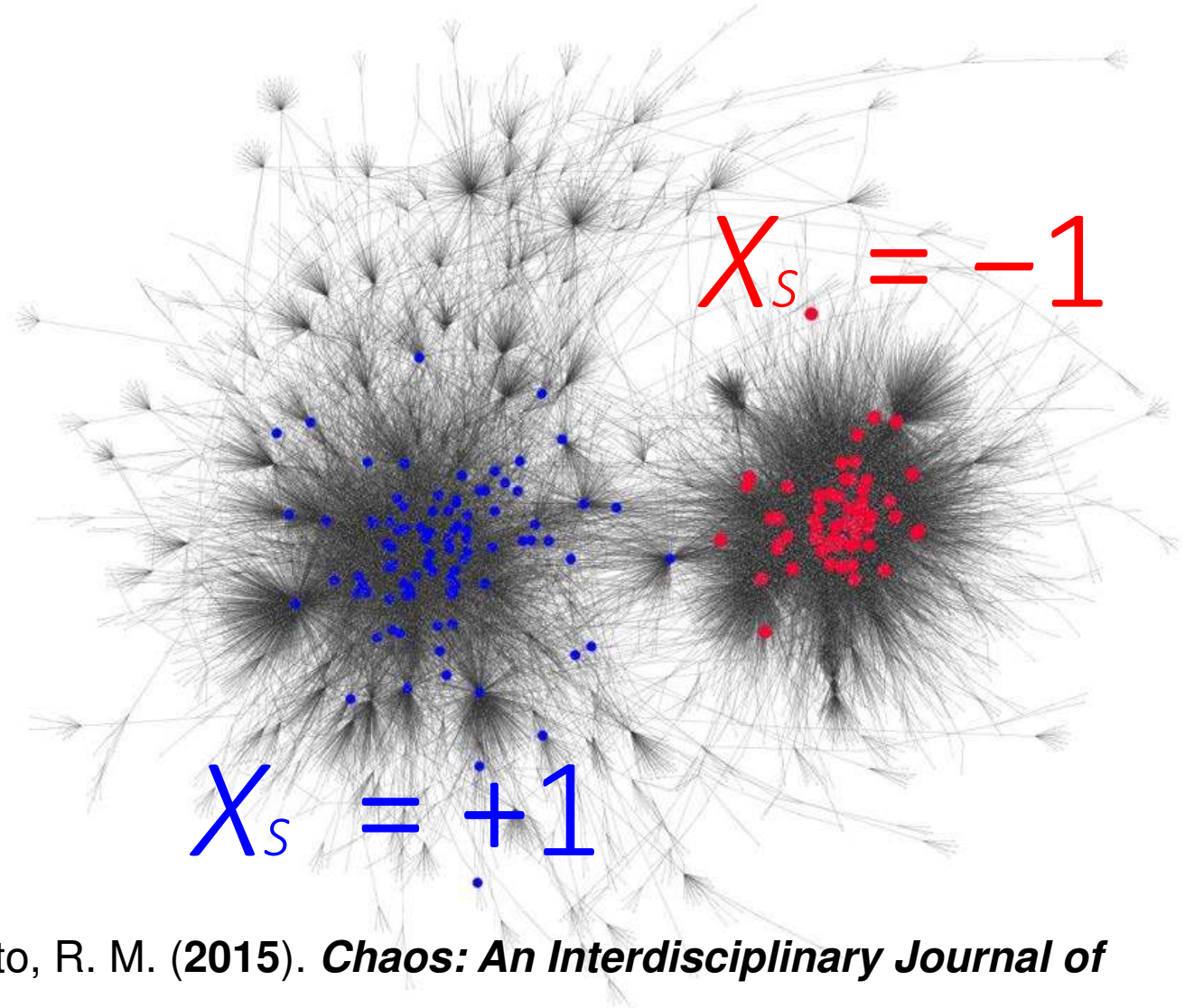
Two type of users:

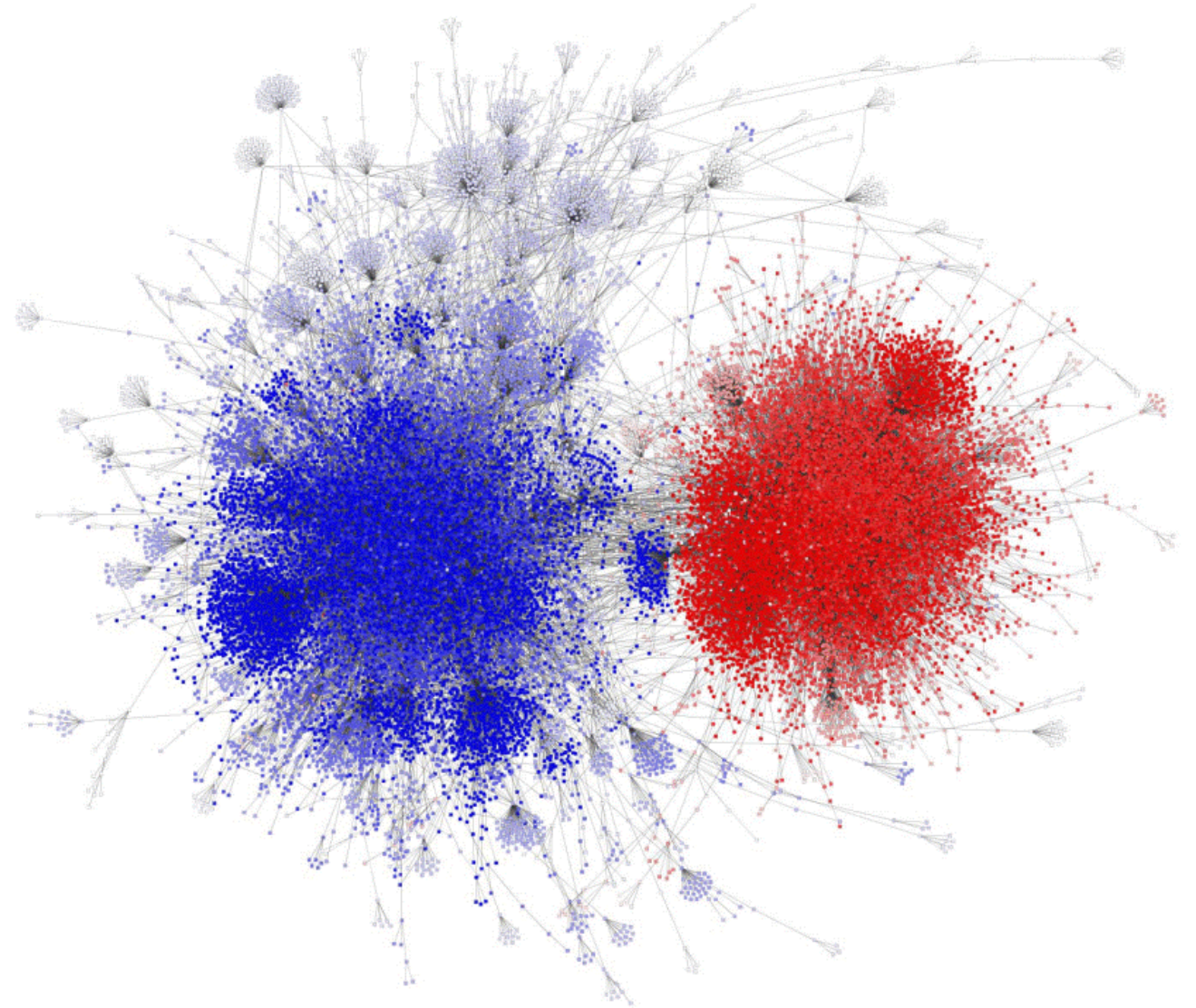
- **Elite:** fixed and antagonist opinions  $X_s$
- **Listeners:** iteratively update their opinions:

$$X_i(t) = \frac{\sum_j A_{ij} X_j(t-1)}{k_i^{out}}$$

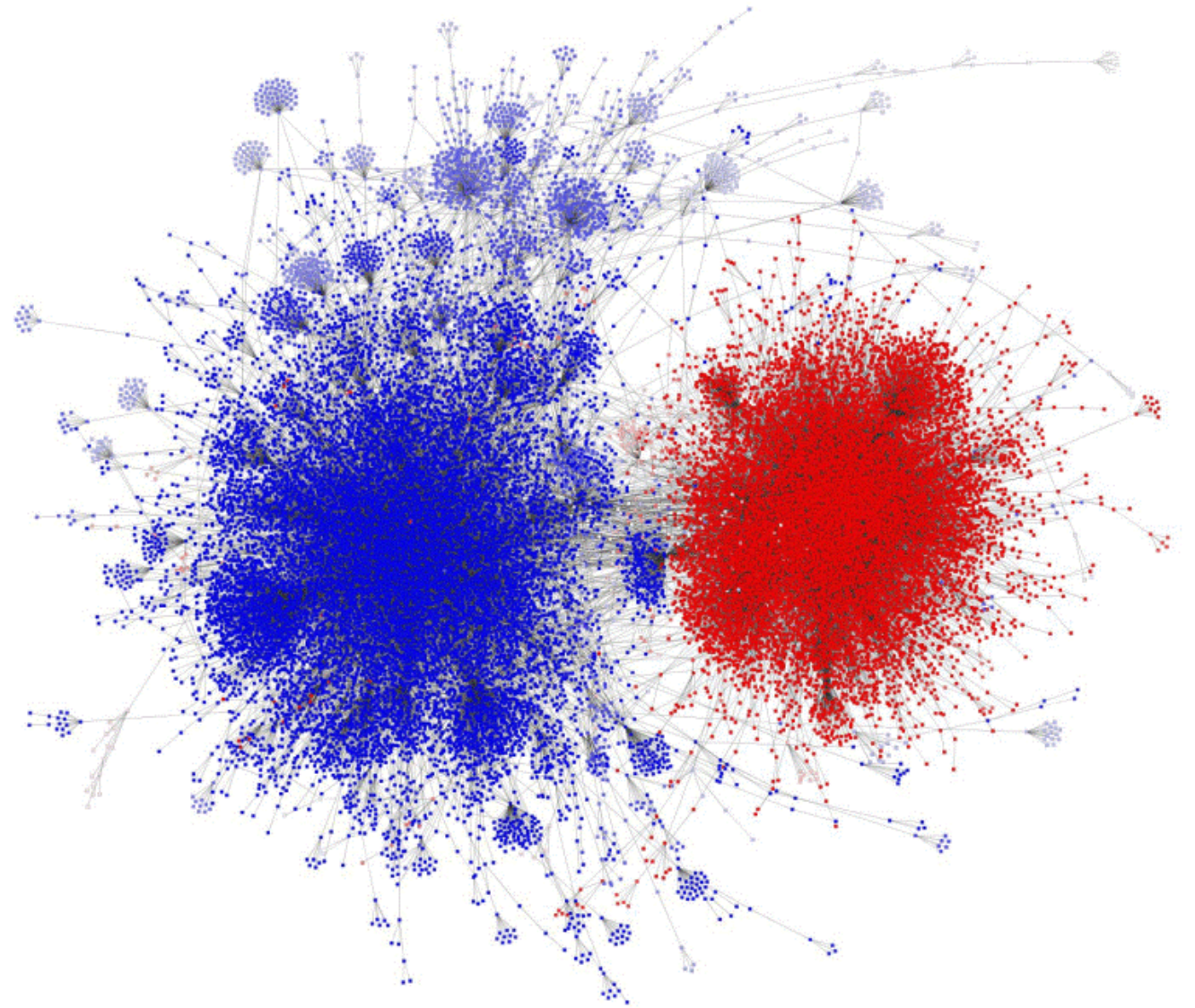
$$X_i(t=0) = 0$$

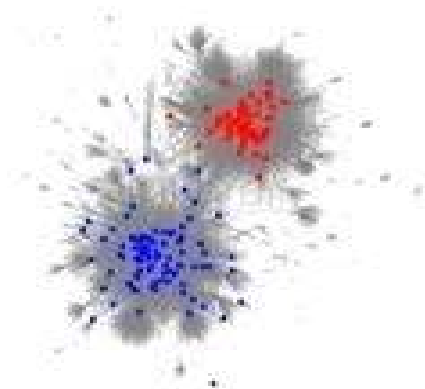
Social Network

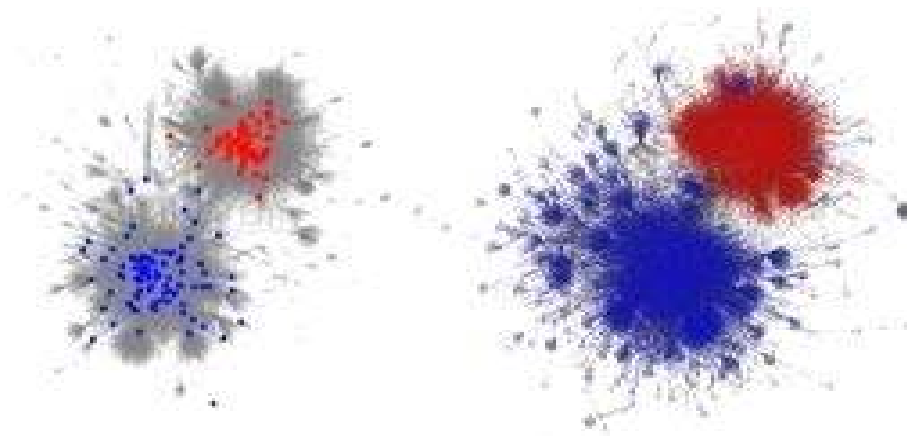




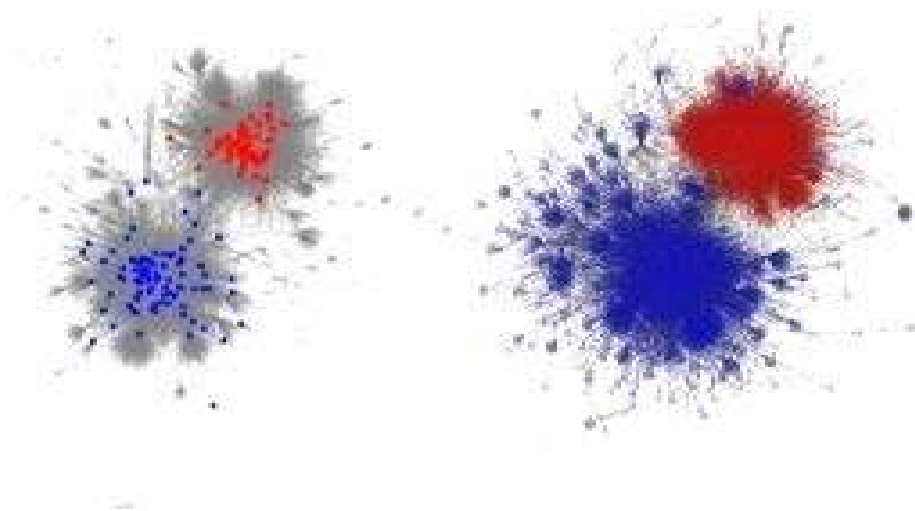




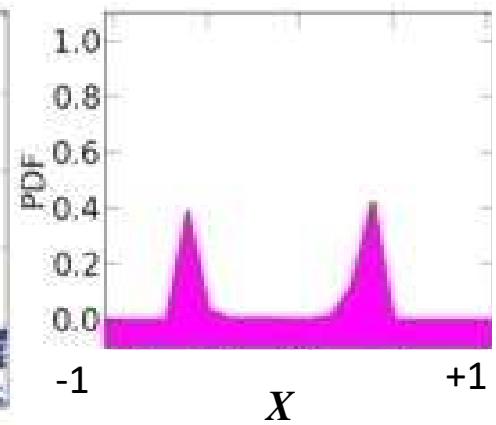
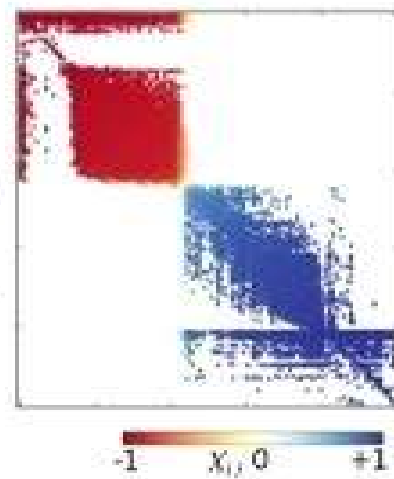




## Distribution of opinions

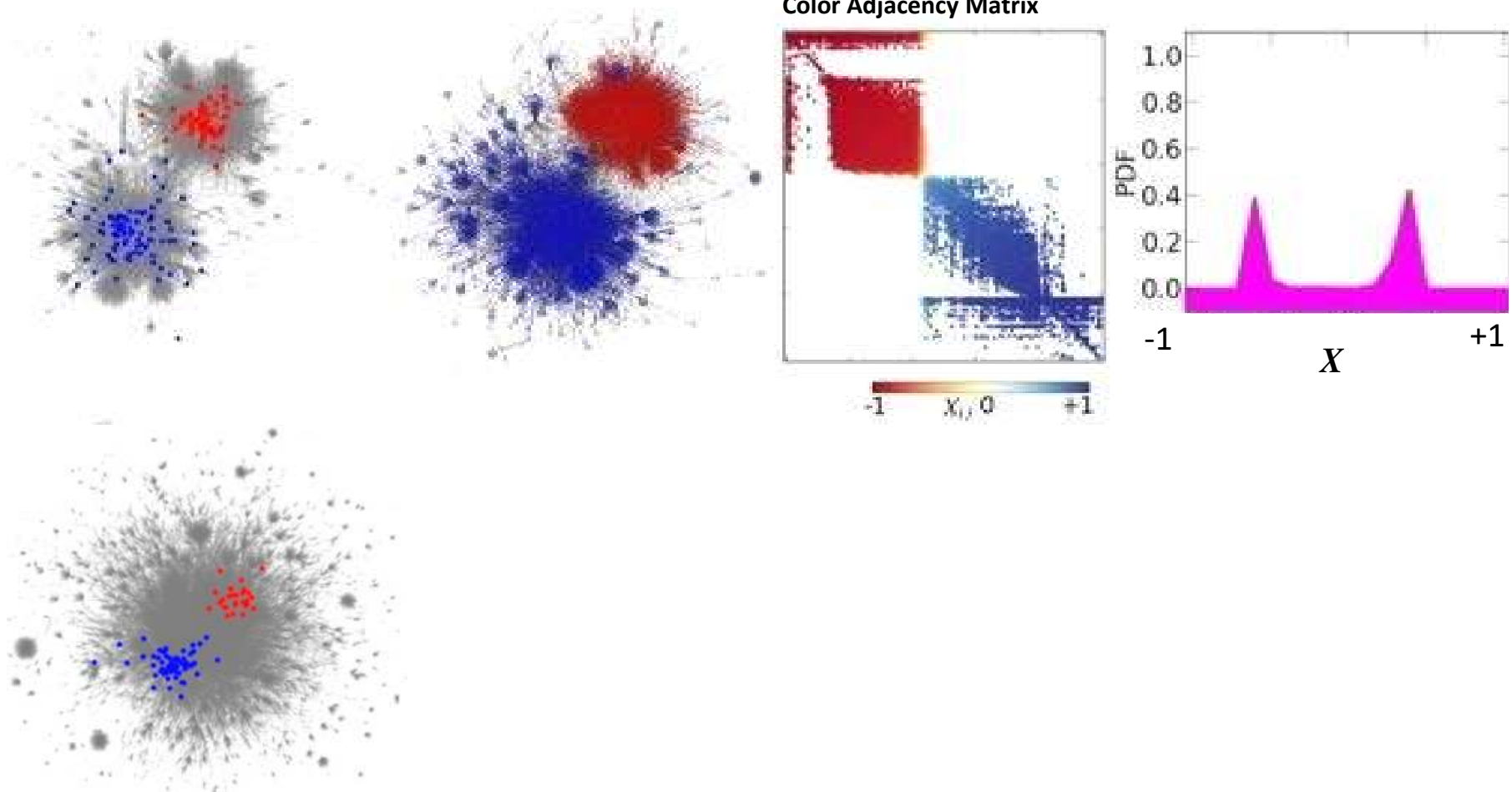


Color Adjacency Matrix

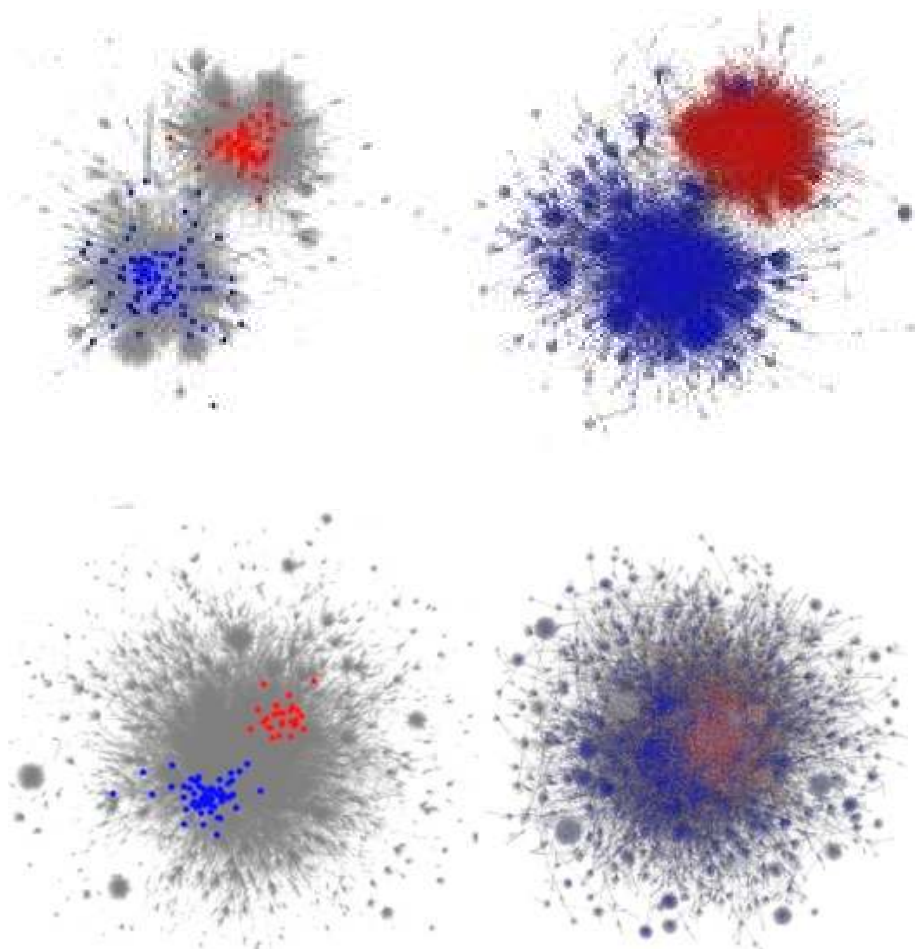




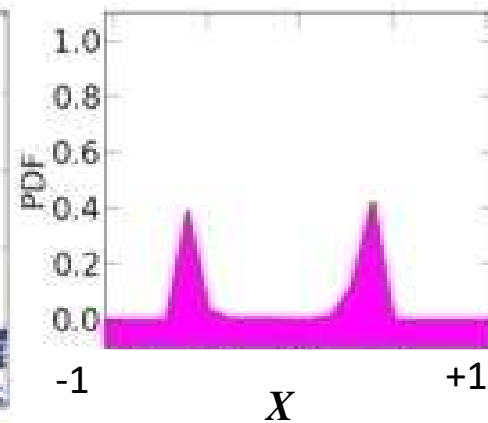
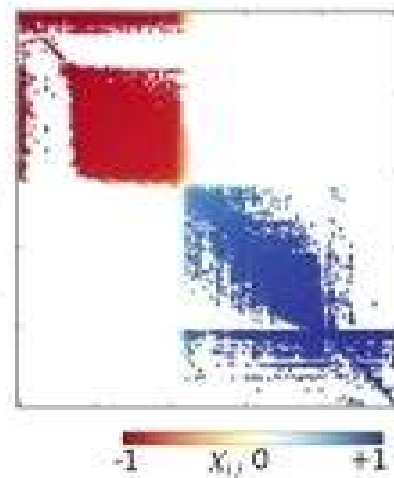
# Distribution of opinions



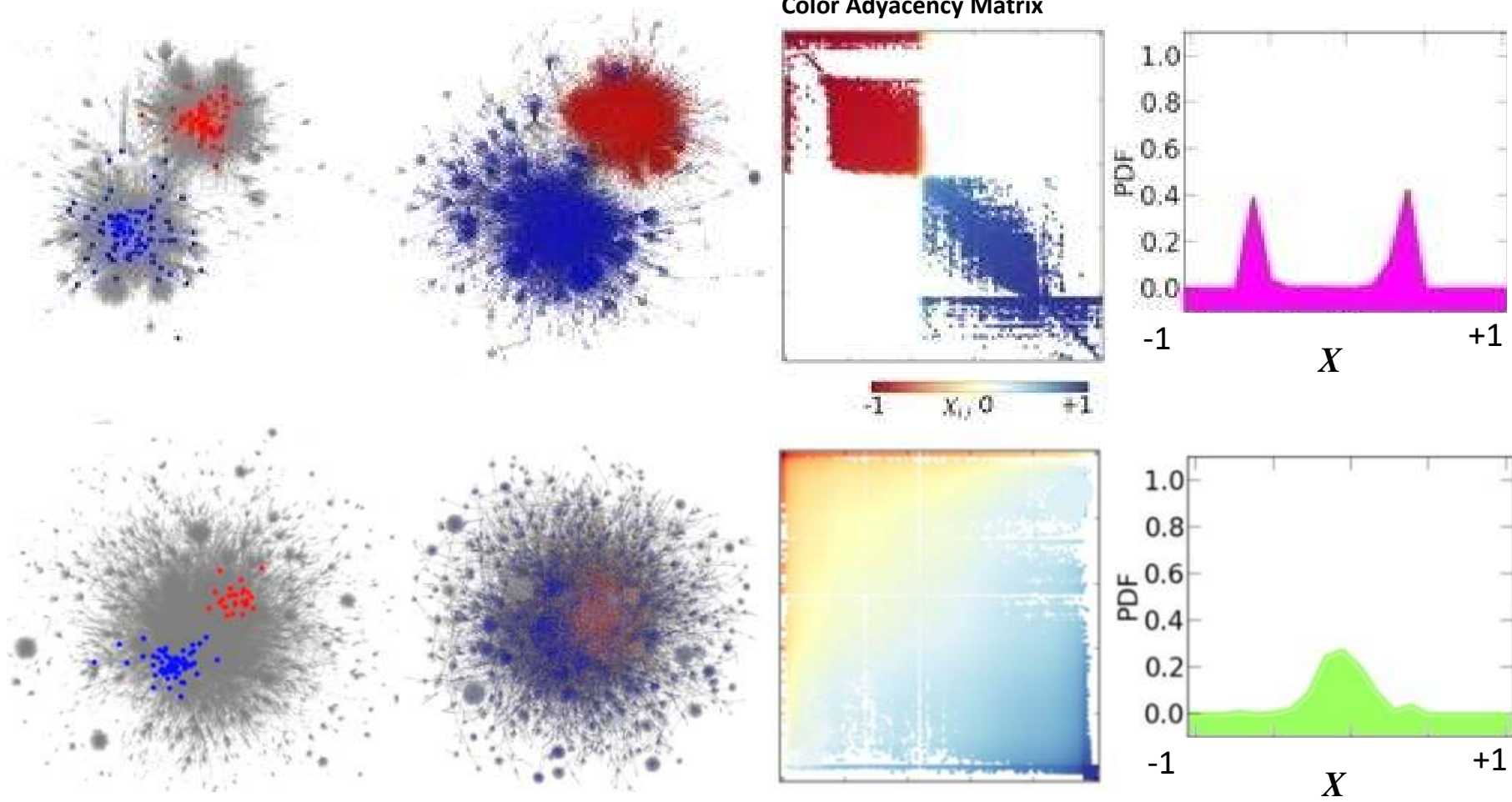
# Distribution of opinions



Color Adyacency Matrix



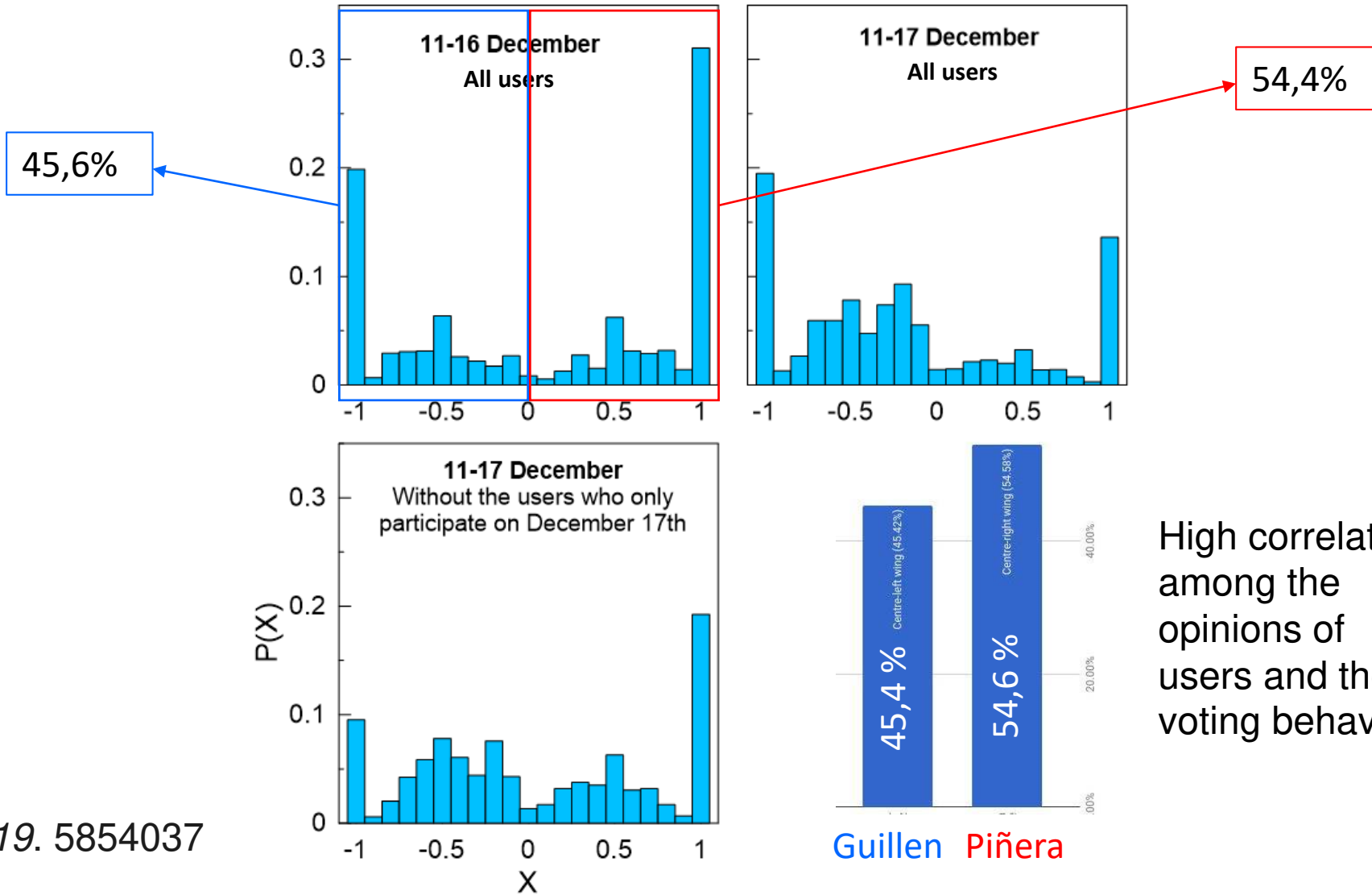
# Distribution of opinions



# Opinion Polarization during a Dichotomous Electoral Process.

## 2nd round 2017 Chilean Presidential Elections

### Opinion distribution

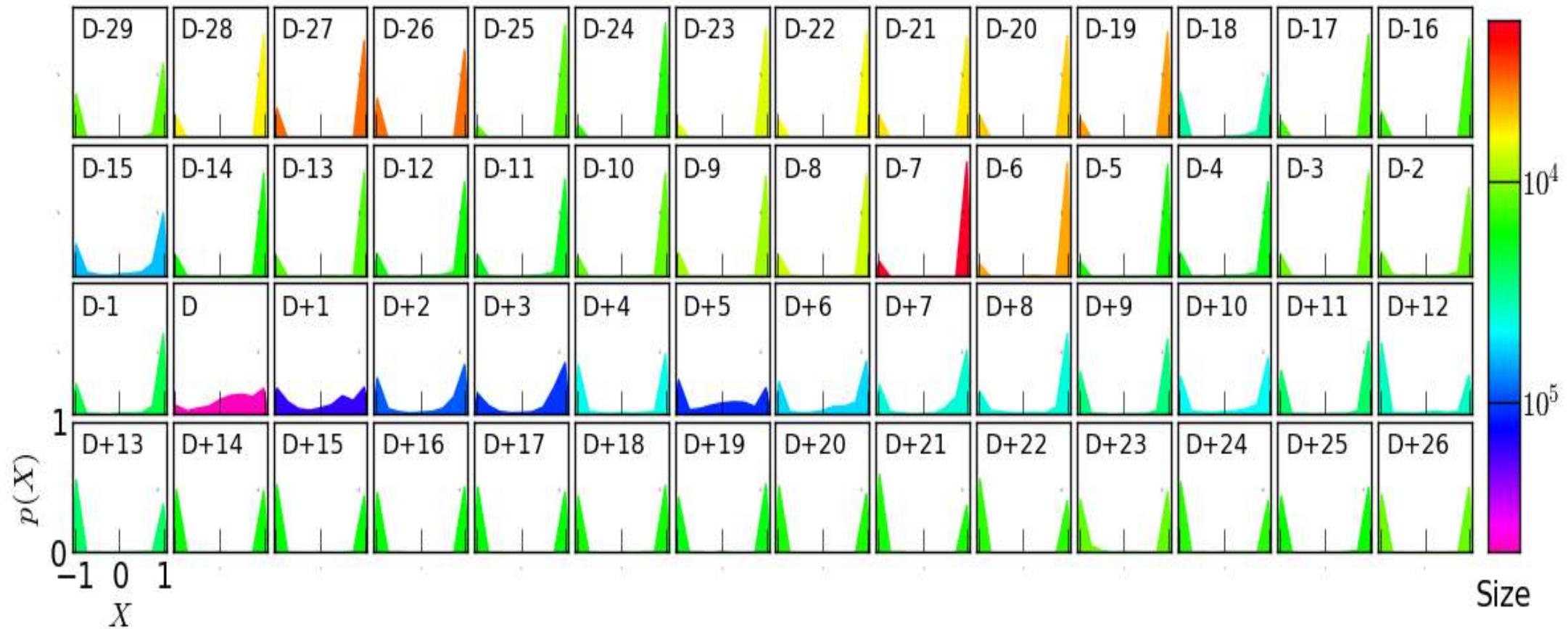




## Results: Opinion distributions

Twitter conversation on the late Venezuelan president, Hugo Chavez

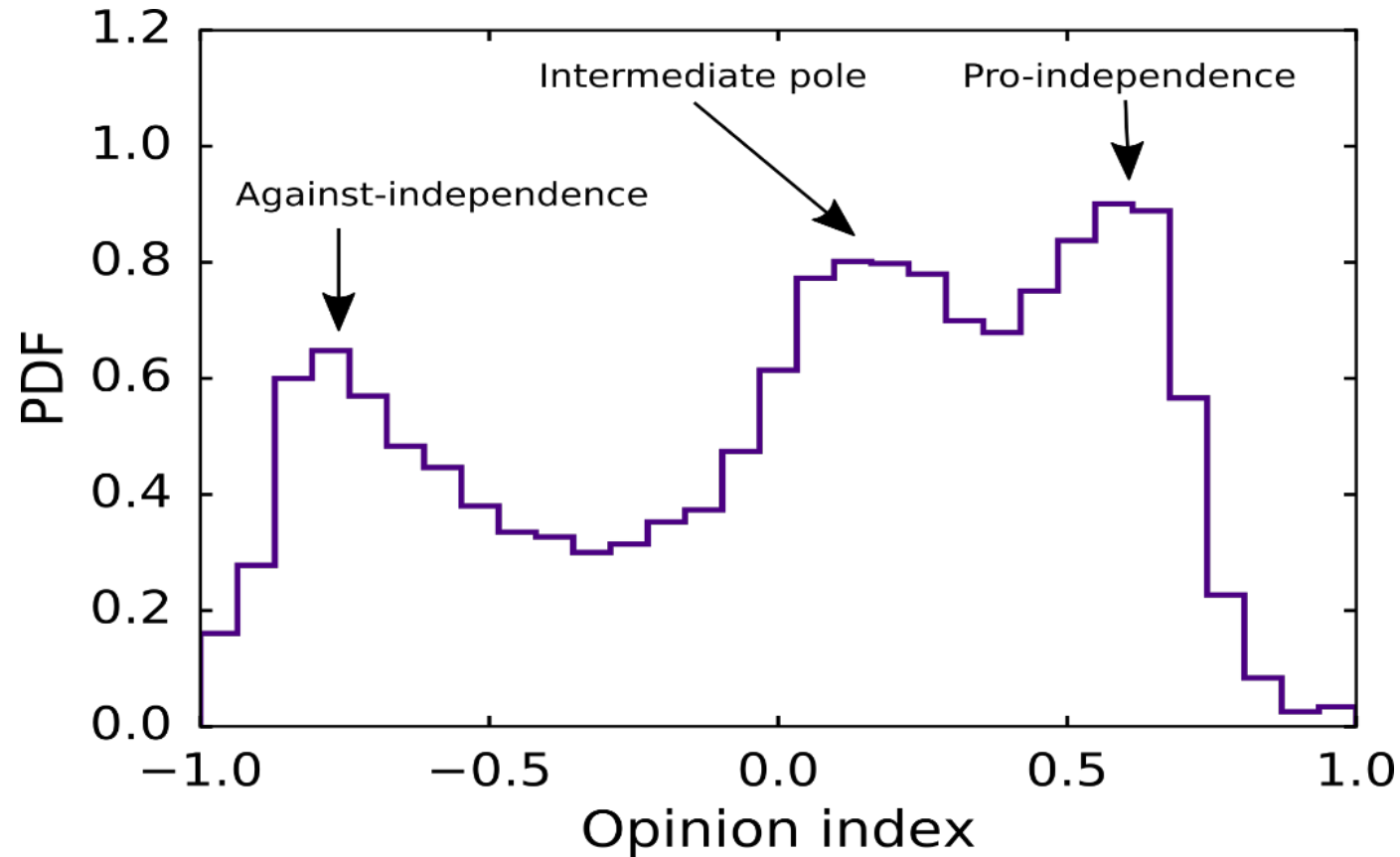
# Time evolution of ideological value Probability Density Functions



Measuring political polarization: Twitter shows the two sides of Venezuela

A. J. Morales, J. Borondo, J. C. Losada, and R. M. Benito, **Chaos 25, 033114 (2015)**

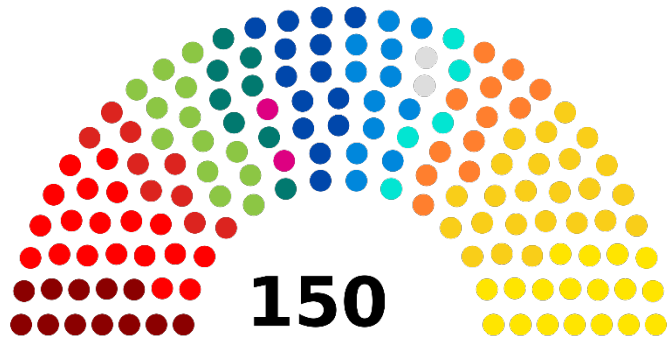
# Twitter conversation about the Catalan independence issue around the 1-0 referendum



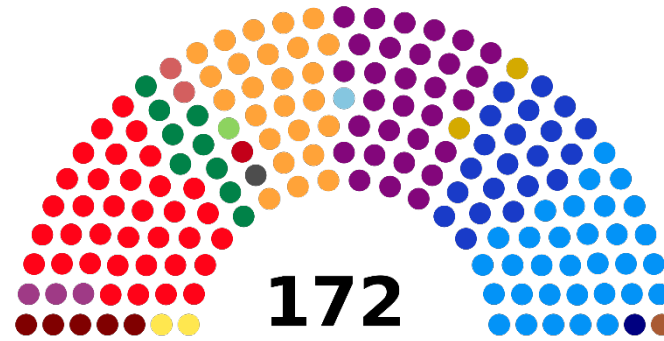
Atienza-Barthelemy, J., Martin-Gutierrez, S., Losada, J. C., Benito, R.M. Relationship between ideology and language in the Catalan Independence context. ***Scientific Reports*, 2019. DOI: 10.1038-s41598-019-53404-x**

# Multipolar systems

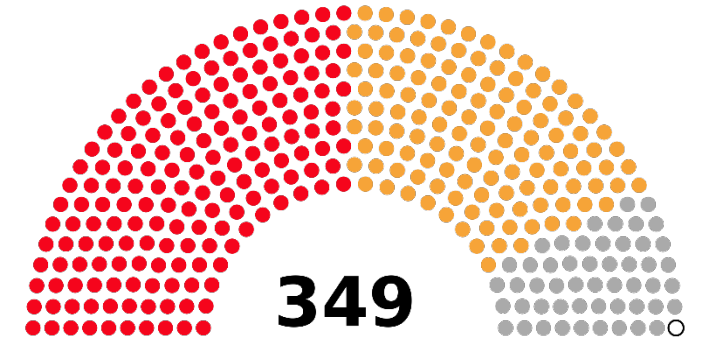
**Belgium**



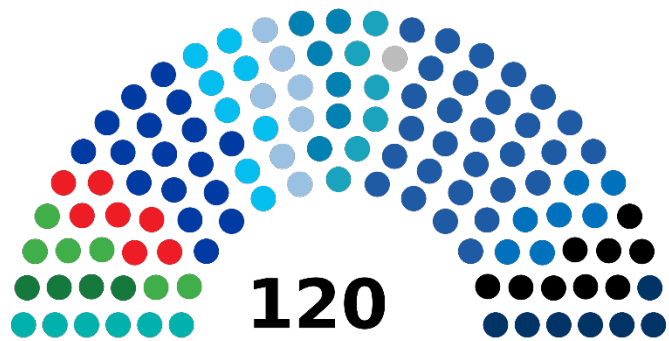
**Colombia**



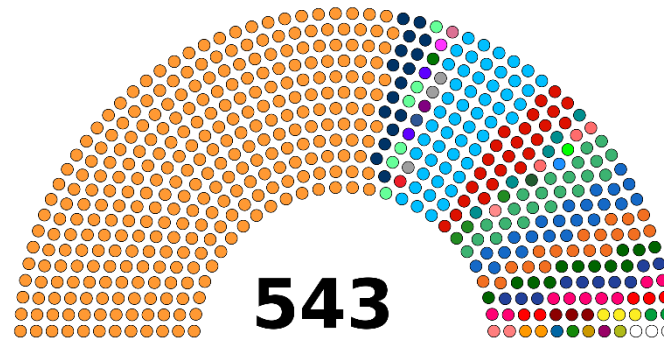
**Kenya**



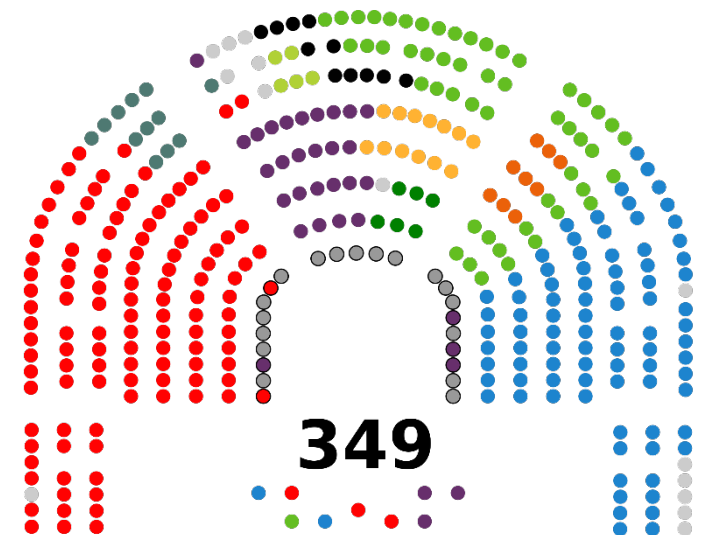
**Israel**

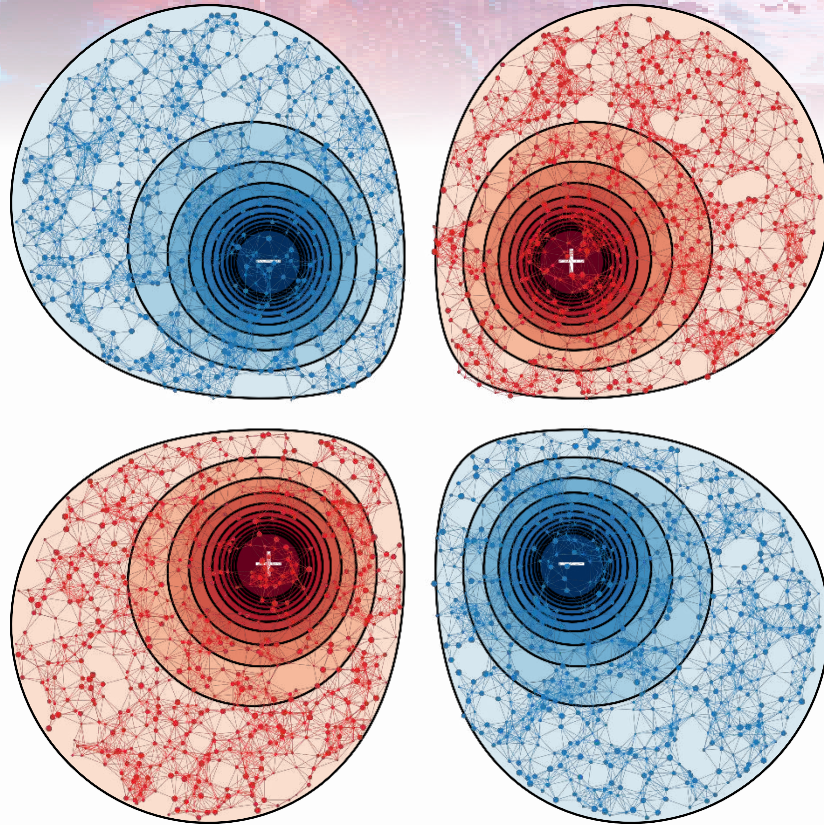


**India**



**Spain**





Modeling a multipolar opinion space

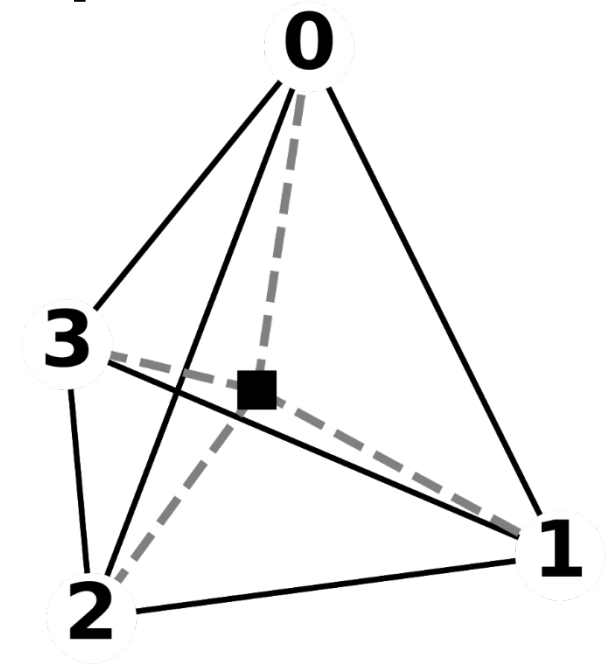
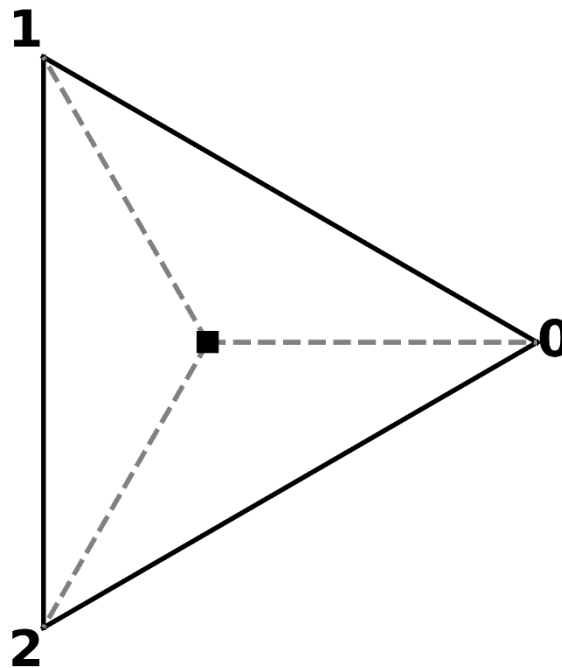


# Pole vectors: The opinion simplex

**Bipolar System**  
space-1D

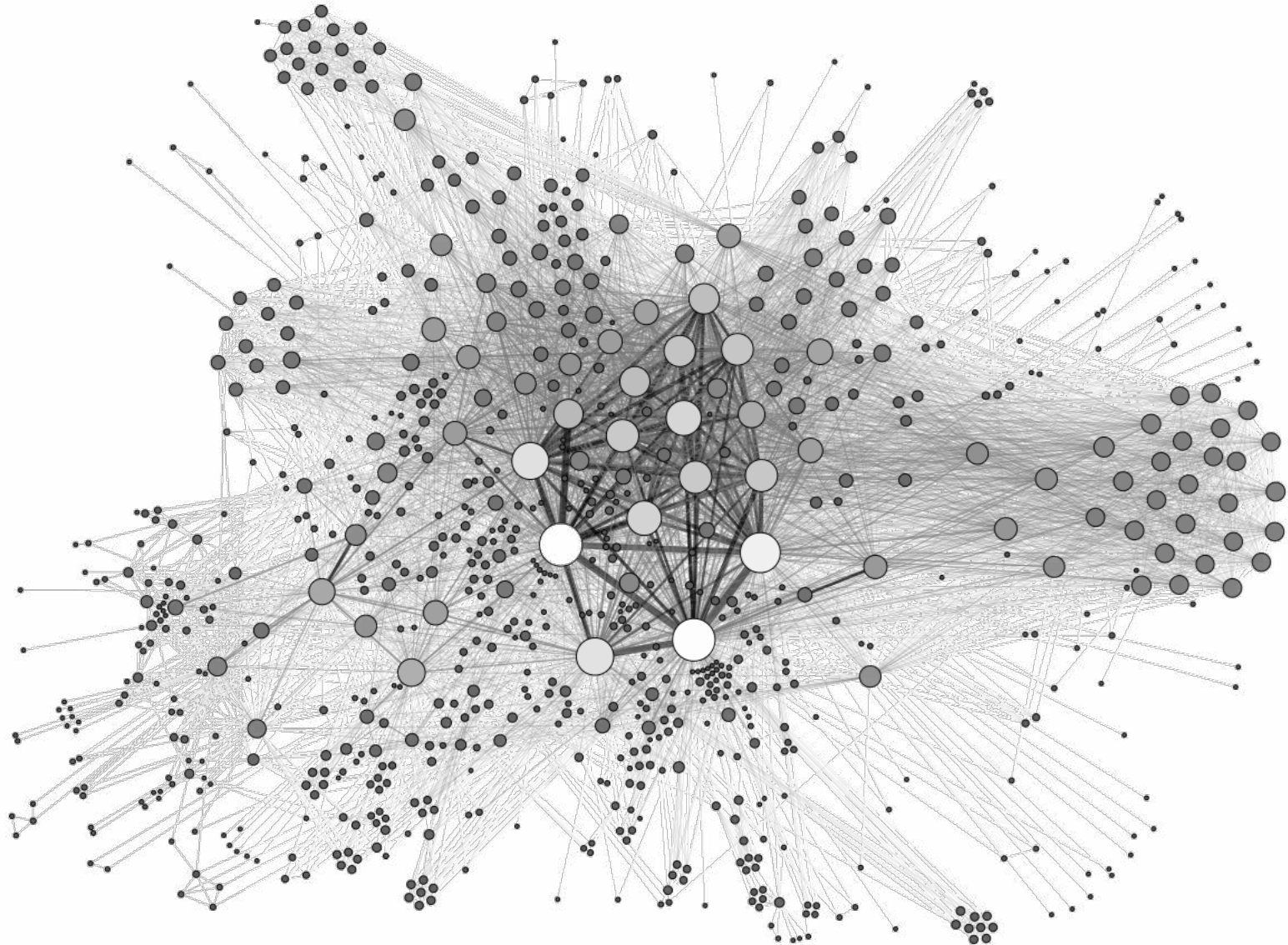
**Tripolar System**  
space-2D

**Quadripolar System**  
space-3D



# Interaction networks: elite and listeners

Retweet  
network

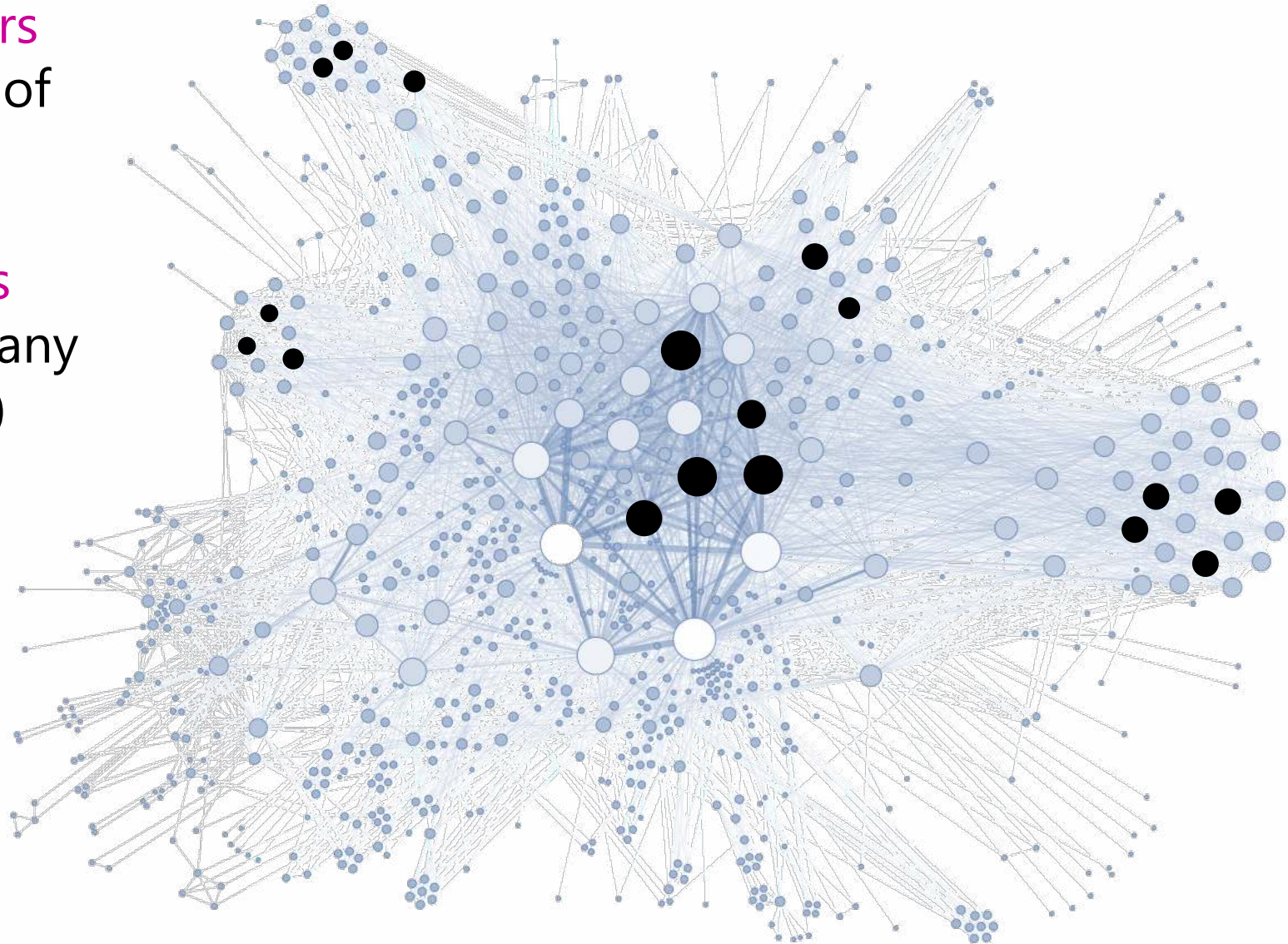




# Interaction networks: elite and listeners

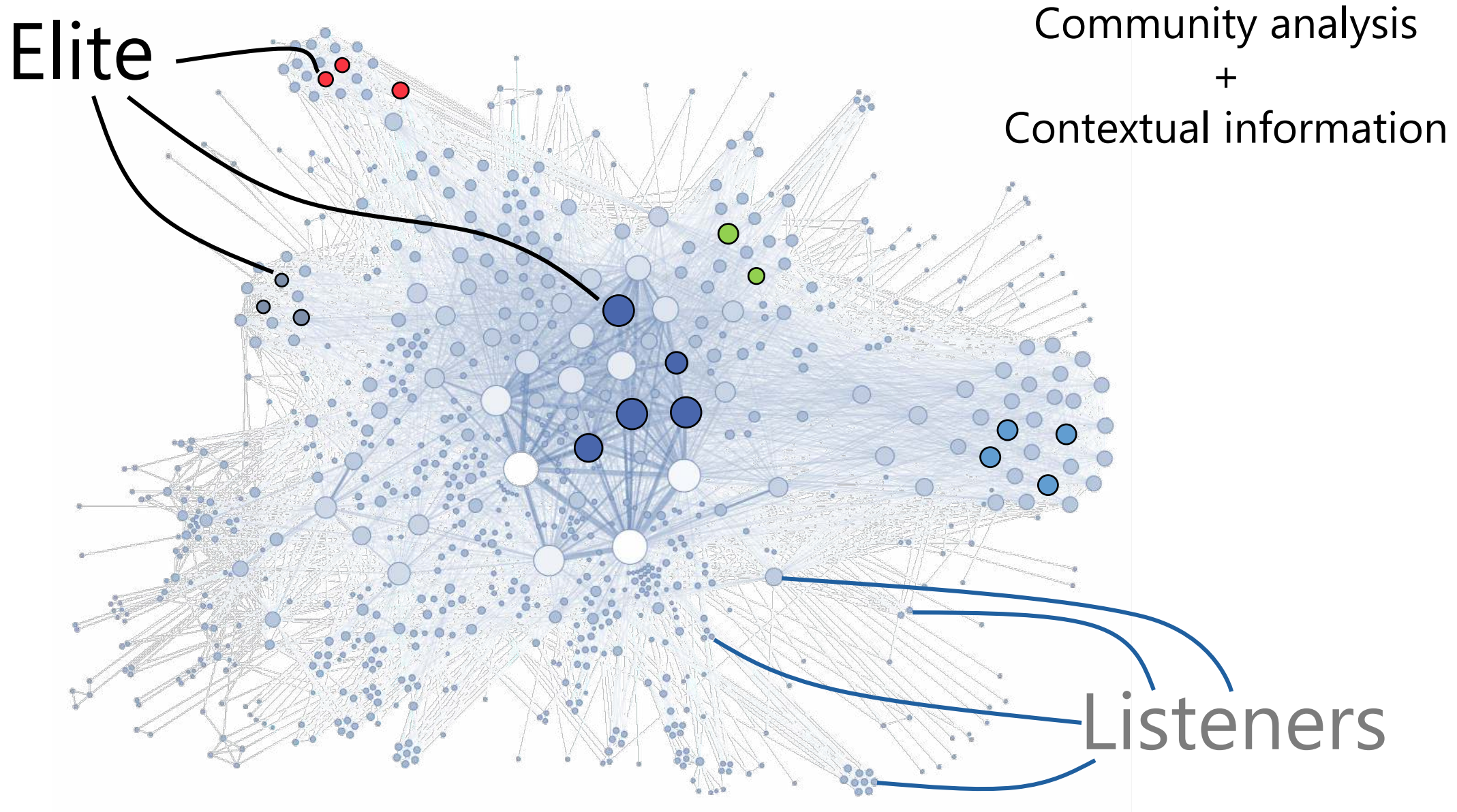
**Influential users**  
(high number of retweets)

**Engaged users**  
(participate many different days)





# Interaction networks: elite and listeners





# Opinion inference: DeGroot model

$\vec{x}_i \rightarrow$  Opinion vector of user  $i$

$A_{ij} \rightarrow$  Adjacency matrix of the network

Replace elite's outgoing links by single self-loop

$$A_{ij}^* = \begin{cases} A_{ij} & \text{if } i \text{ is Listener} \\ \delta_{ij} & \text{if } i \text{ is Elite} \end{cases}$$

Iteratively average the opinion of the neighbors

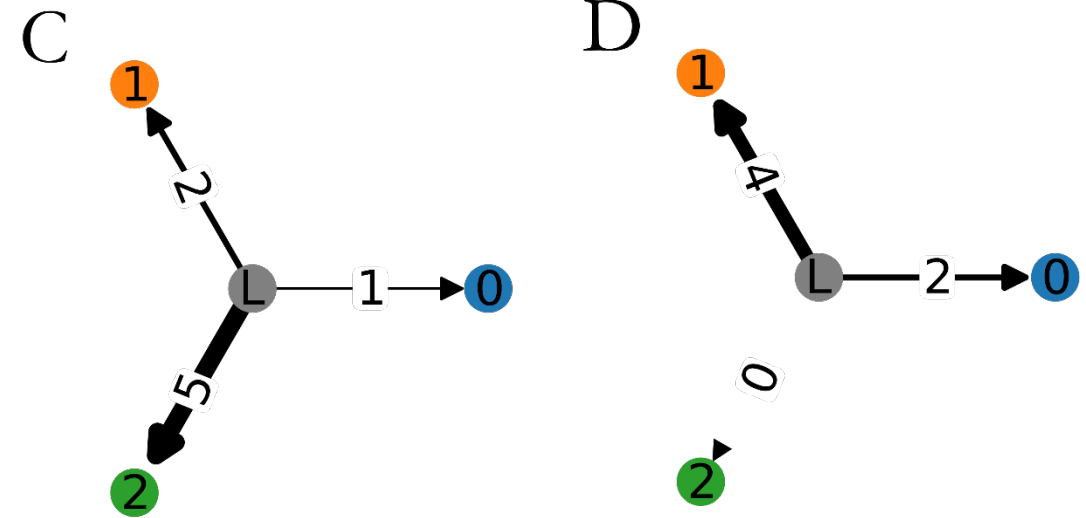
$$\vec{x}_i(t = 0) = \begin{cases} \vec{0} & \text{if } i \text{ is Listener} \\ \vec{x}_p & \text{if } i \text{ is Elite} \end{cases}$$

Vertex of the opinion simplex

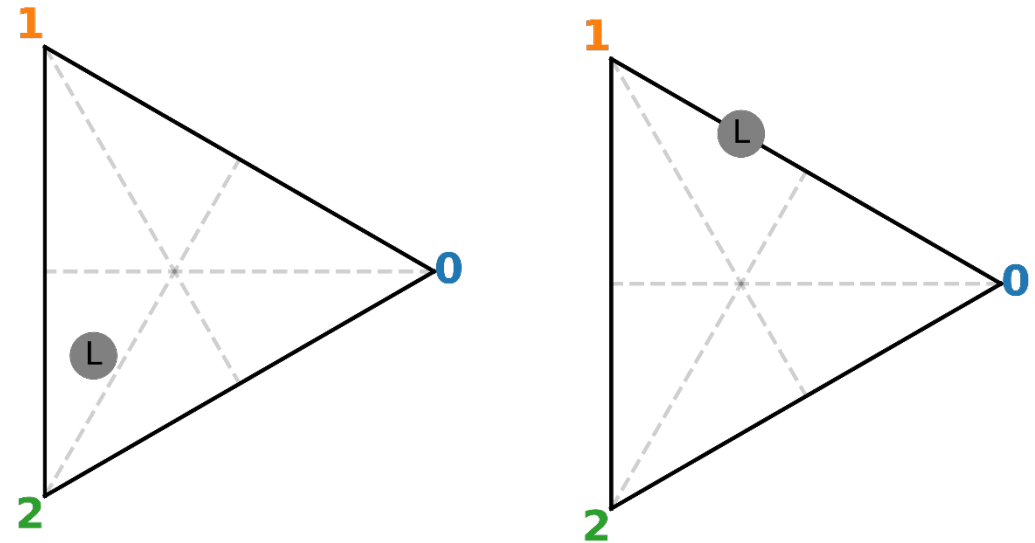
$$\vec{x}_i(t) = \frac{\sum_j A_{ij}^* \vec{x}_j(t-1)}{\sum_j A_{ij}^*}$$

# Example of a tripolar system

Network →



Opinion space →



# Example of a quadripolar system





# MEASURE OF POLARIZATION IN OPINION DISTRIBUTIONS:

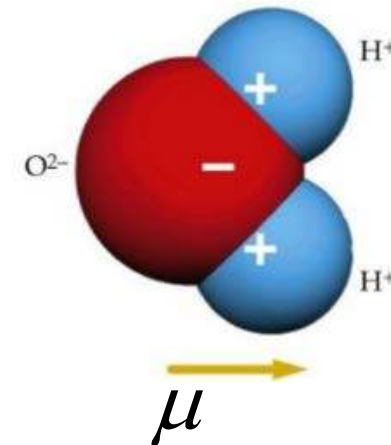
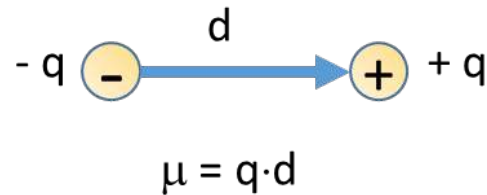
## The Polarization Index

# The Polarization Index

Based on:

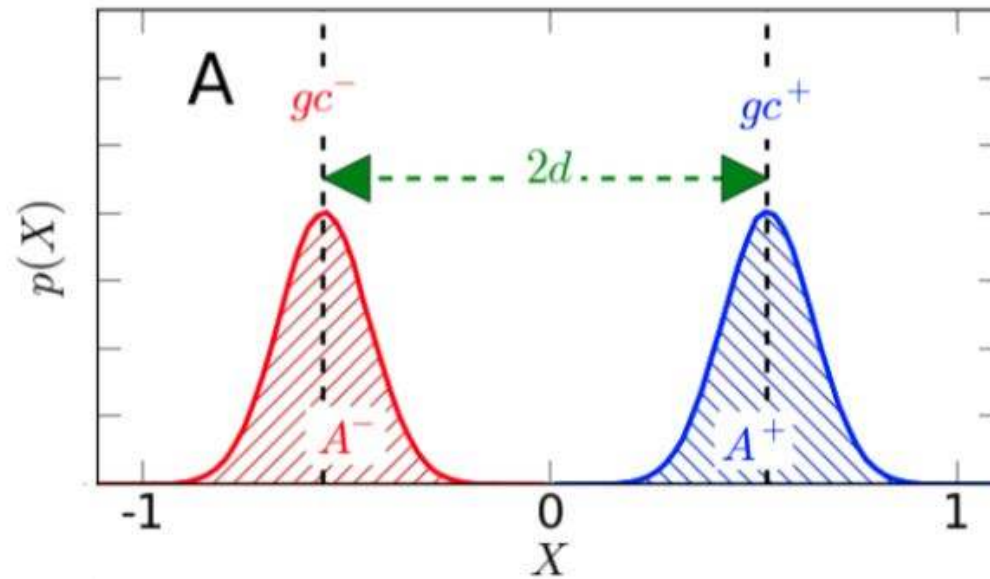
- **A Population is Perfectly Polarized:**  
Divided in **two groups** of the **same size** and **opposite opinions**

- **Dipolar Moment**

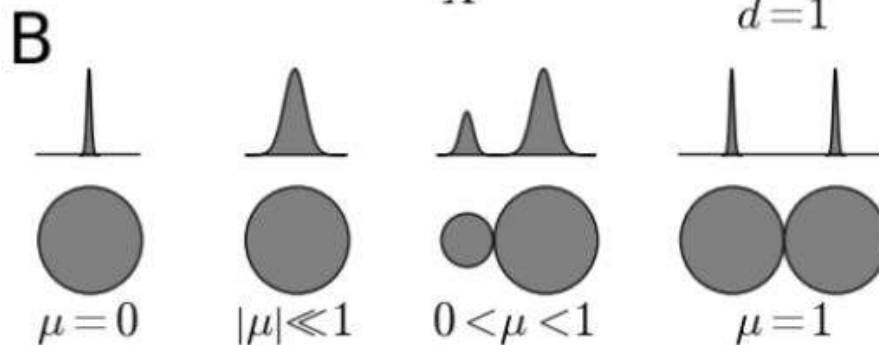
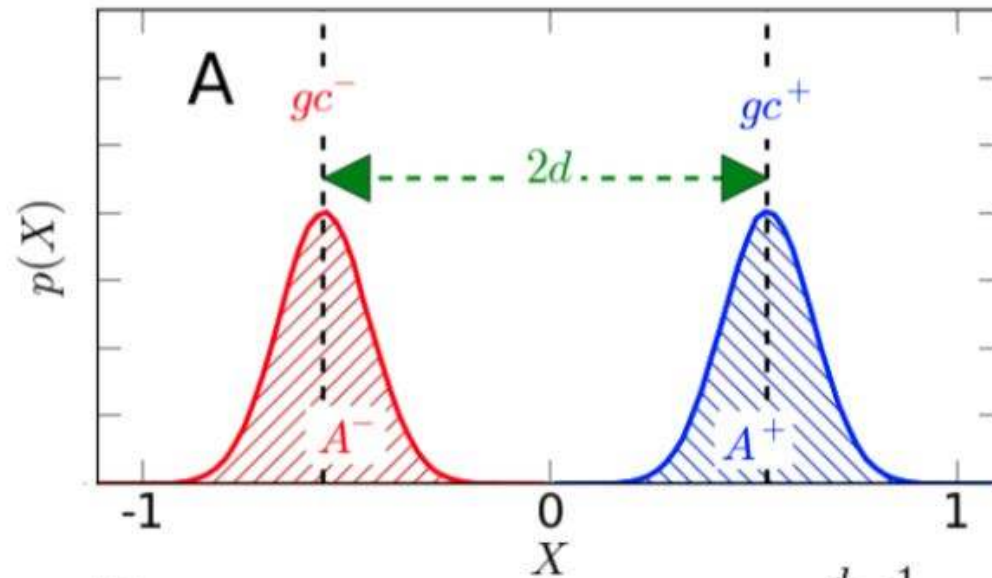




**A Population is Perfectly Polarized:** Divided in two groups of the same size and opposite opinions



A Population is **Perfectly Polarized**: Divided in two groups of the **same size** and **opposite opinions**



Polarization indices

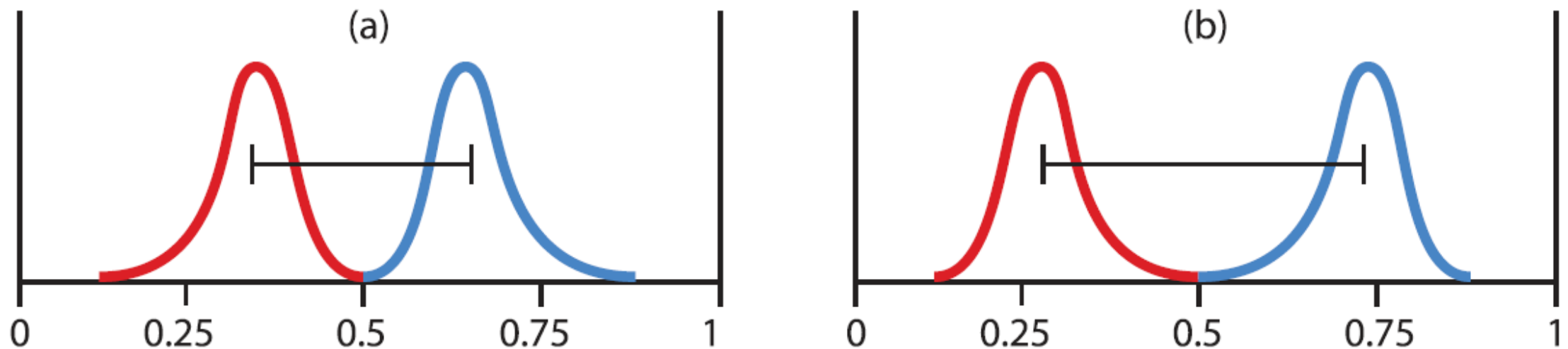
$$\mu = (1 - \Delta A)d$$

$$d = \frac{|gc^+ - gc^-|}{|X_{max} - X_{min}|}$$

$$\Delta A = |A^+ - A^-|$$

# Measures of Polarization in multipolar systems: The covariance and PCA

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

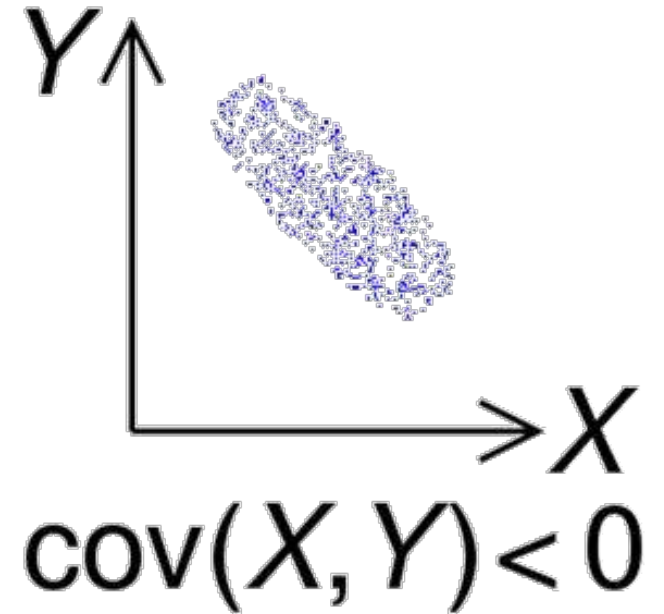
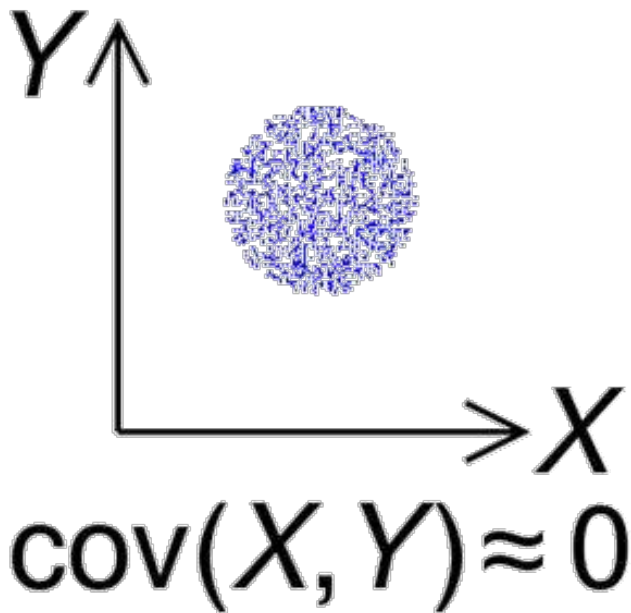
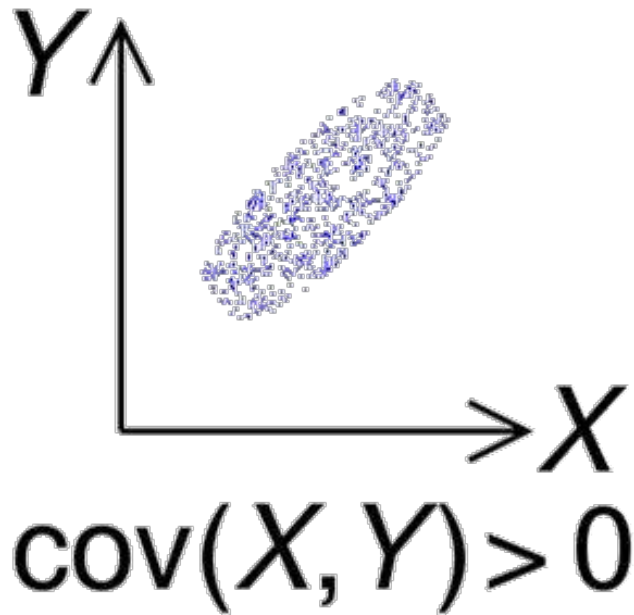


Aaron Bramson, Patrick Grim, Daniel J. Singer, Steven Fisher, William Berger, Graham Sack & Carissa Flocken (2016) Disambiguation of social polarization concepts and measures, *The Journal of Mathematical Sociology*, 40:2, 80-111, DOI: [10.1080/0022250X.2016.1147443](https://doi.org/10.1080/0022250X.2016.1147443)

# Measures of Polarization in multipolar systems: The covariance and PCA

Multidimensional generalization of the variance: **the covariance**

$$\text{Cov}(X_i, X_j) = E[(X_i - E[X_i])(X_j - E[X_j])]$$



# Measures of Polarization in multipolar systems: The covariance and PCA

$$\text{Cov}(X_i, X_j) = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])]$$

$$\text{Cov}[\vec{X}, \vec{X}] = \begin{bmatrix} \mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_1 - \mathbb{E}[X_1])] & \mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2])] & \cdots & \mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_n - \mathbb{E}[X_n])] \\ \mathbb{E}[(X_2 - \mathbb{E}[X_2])(X_1 - \mathbb{E}[X_1])] & \mathbb{E}[(X_2 - \mathbb{E}[X_2])(X_2 - \mathbb{E}[X_2])] & \cdots & \mathbb{E}[(X_2 - \mathbb{E}[X_2])(X_n - \mathbb{E}[X_n])] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[(X_n - \mathbb{E}[X_n])(X_1 - \mathbb{E}[X_1])] & \mathbb{E}[(X_n - \mathbb{E}[X_n])(X_2 - \mathbb{E}[X_2])] & \cdots & \mathbb{E}[(X_n - \mathbb{E}[X_n])(X_n - \mathbb{E}[X_n])] \end{bmatrix}$$



# Measures of Polarization in multipolar systems: The covariance and PCA

The **trace of the covariance matrix** :

as a measure of *multidimensional variance*, usually called *total variation* ( $TV$ ).

- the **maximum** attainable  $TV$  is **1**,

so this metric is **normalized** by design when the distance between the barycenter of the opinion simplex and the poles is  $u = 1$ ,

- **The maximum  $TV$  is achieved when:**

- there are only **extreme opinions**
- they are **uniformly distributed** among all the poles.

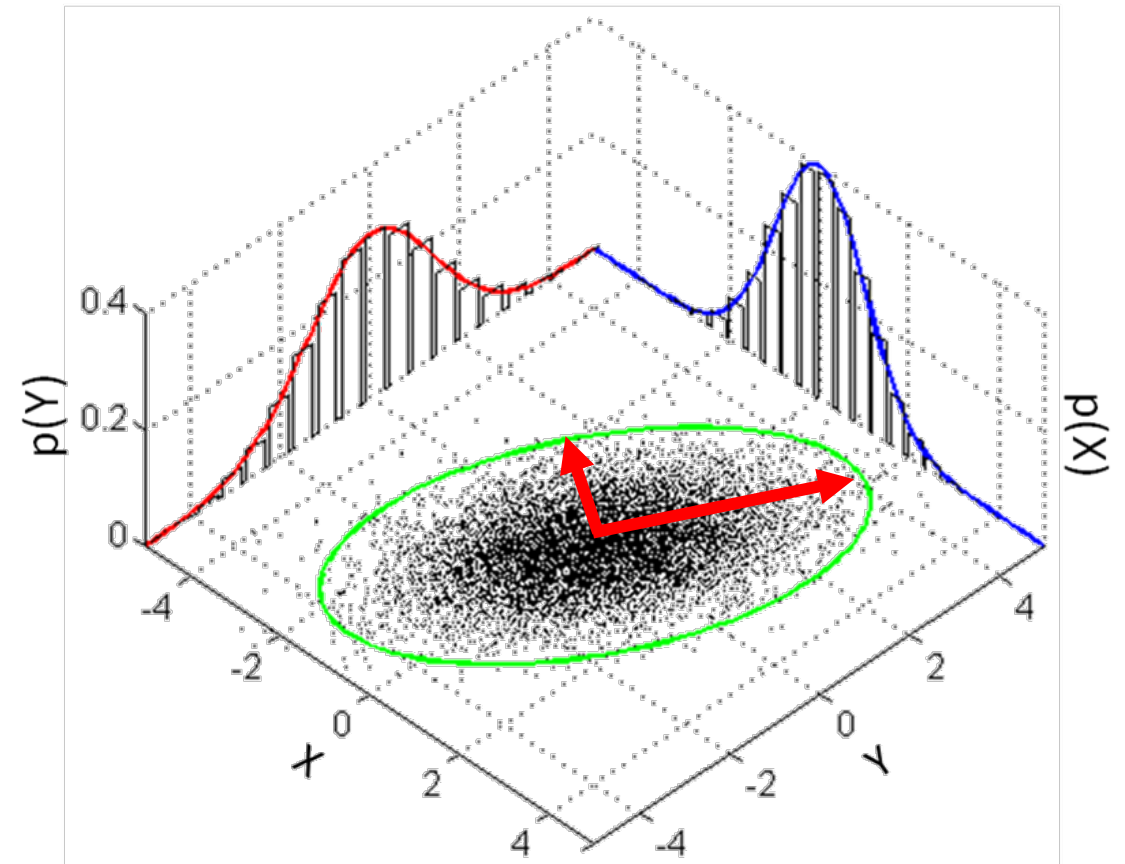
- Therefore, the  $TV$  can be used as a measure of **global polarization** combining the aspects:

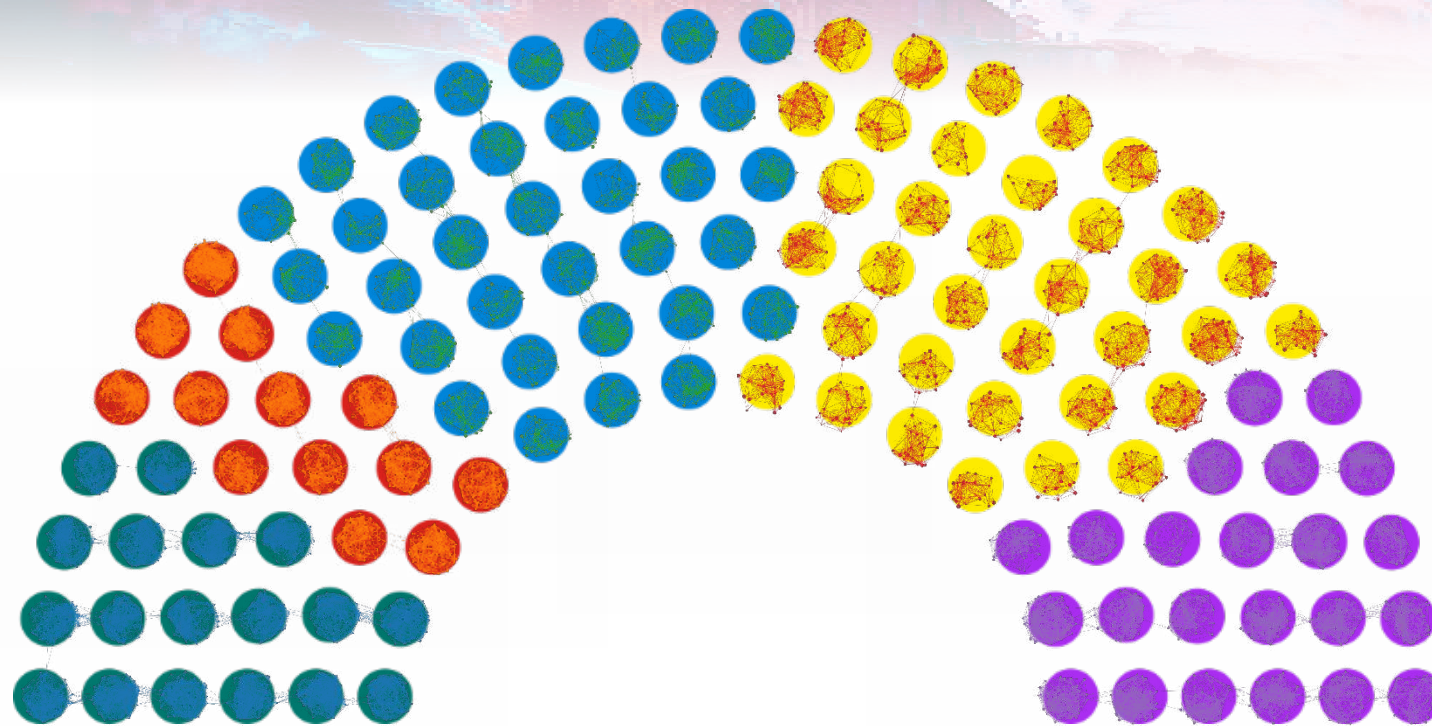
of *opinion extremeness* and *community fragmentation*.

It not only measures how extreme the opinions are but also how evenly is the population divided into the considered factions

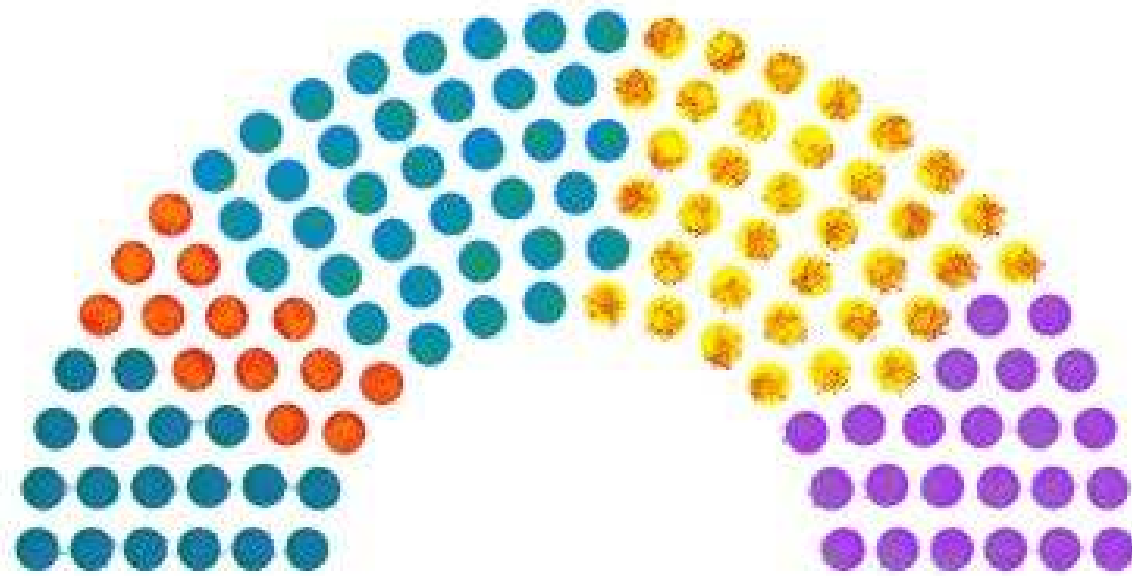
# Measures of Polarization in multipolar systems: The covariance and PCA

- To characterize and **quantify pole alignment** we compute the eigendecomposition of the covariance matrix, (*Principal Component Analysis* (PCA)).
  - The **eigenvectors** (or Principal Components — PCs) form an orthogonal basis of the opinion space
  - Each **eigenvalue** corresponds to the projected variance along the direction defined by the eigenvector.
- The eigenvector with the largest eigenvalue (**first PC**): corresponds to the **direction of maximum variance**: direction of **maximum polarization**.





Empirical study of multipolar systems



Empirical study of multipolar systems

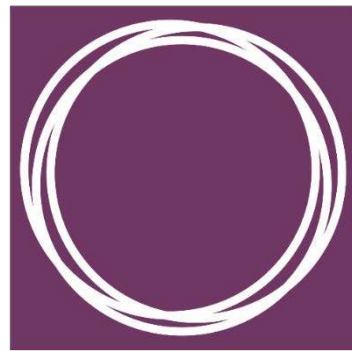
# Quadripolar system: Spanish elections of December 2015



**0 - Christian conservatism**



**1 - Social democratic**



**PODEMOS**  
**2 - Left-wing**  
**populism**



**3 - Liberalism**

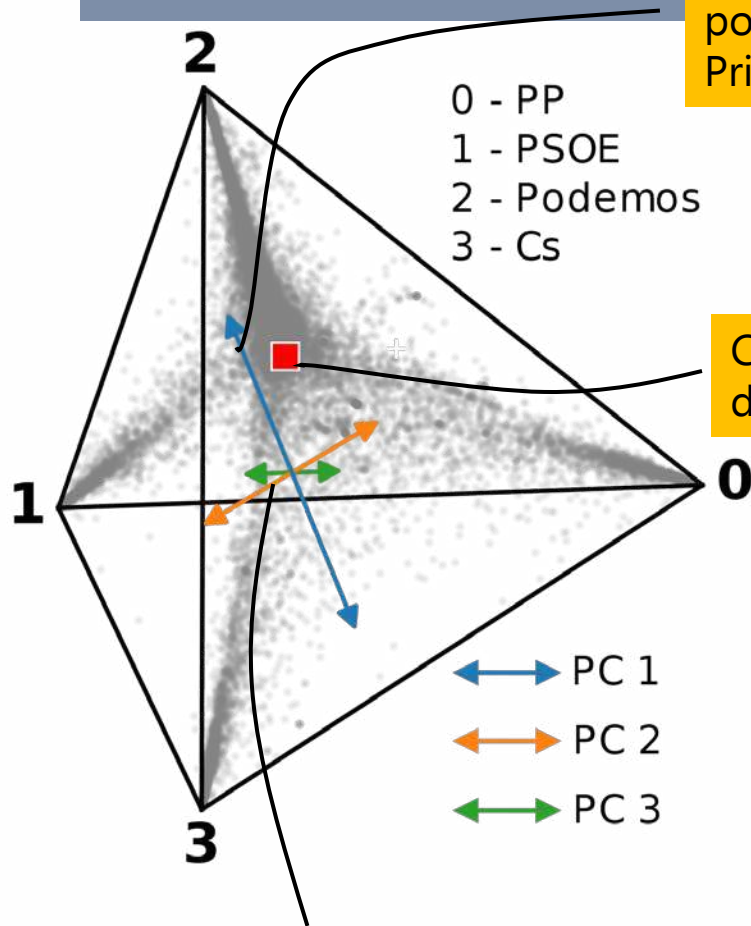


# Quadripolar system: Spanish elections 2015

Direction of maximum polarization (First Principal Component)

- 0 - PP
- 1 - PSOE
- 2 - Podemos
- 3 - Cs

Center of mass of the distribution (average)

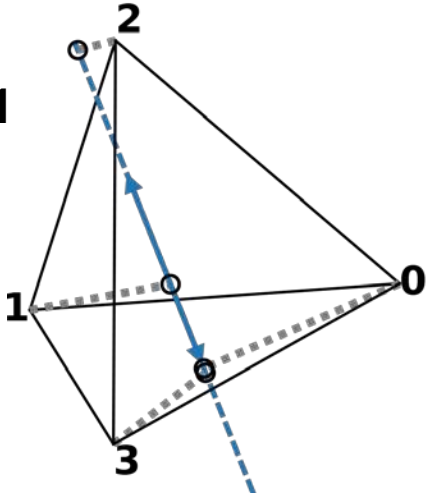


- PC 1
- PC 2
- PC 3

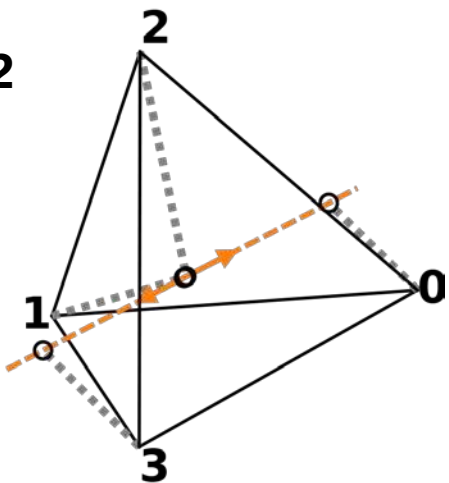
Neutral point (center)

# Quadripolar system: Spanish elections 2015

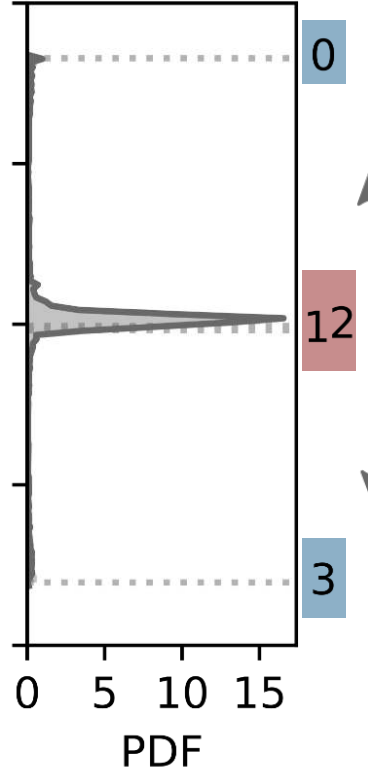
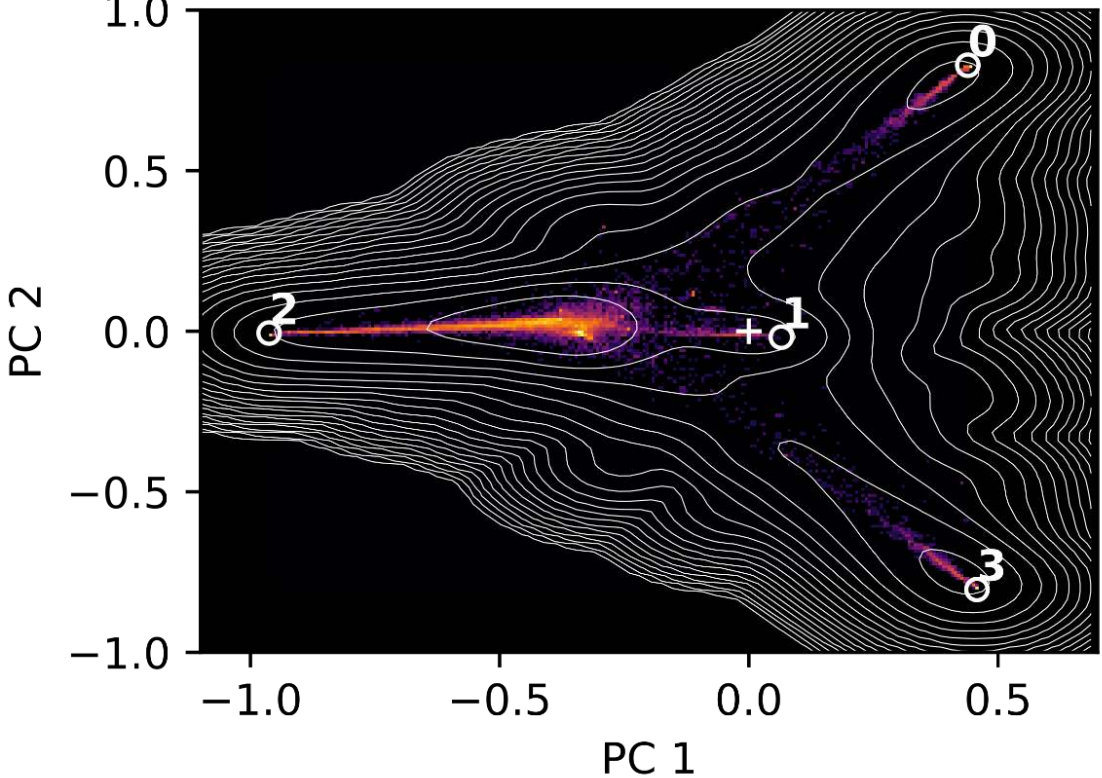
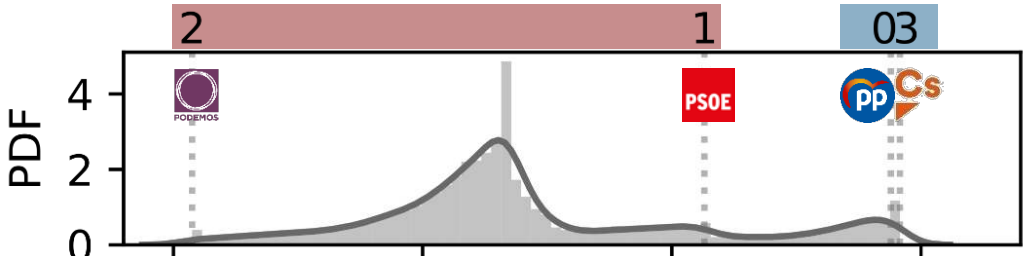
Principal Component 1



Principal Component 2



Left-wing parties      Right-wing parties



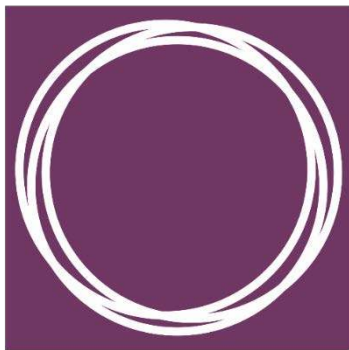
# Pentapolar system: Spanish elections of April 2019



**0 - Christian conservatism**



**1 - Social democratic**



**PODEMOS**  
**2 - Left-wing**  
**populism**

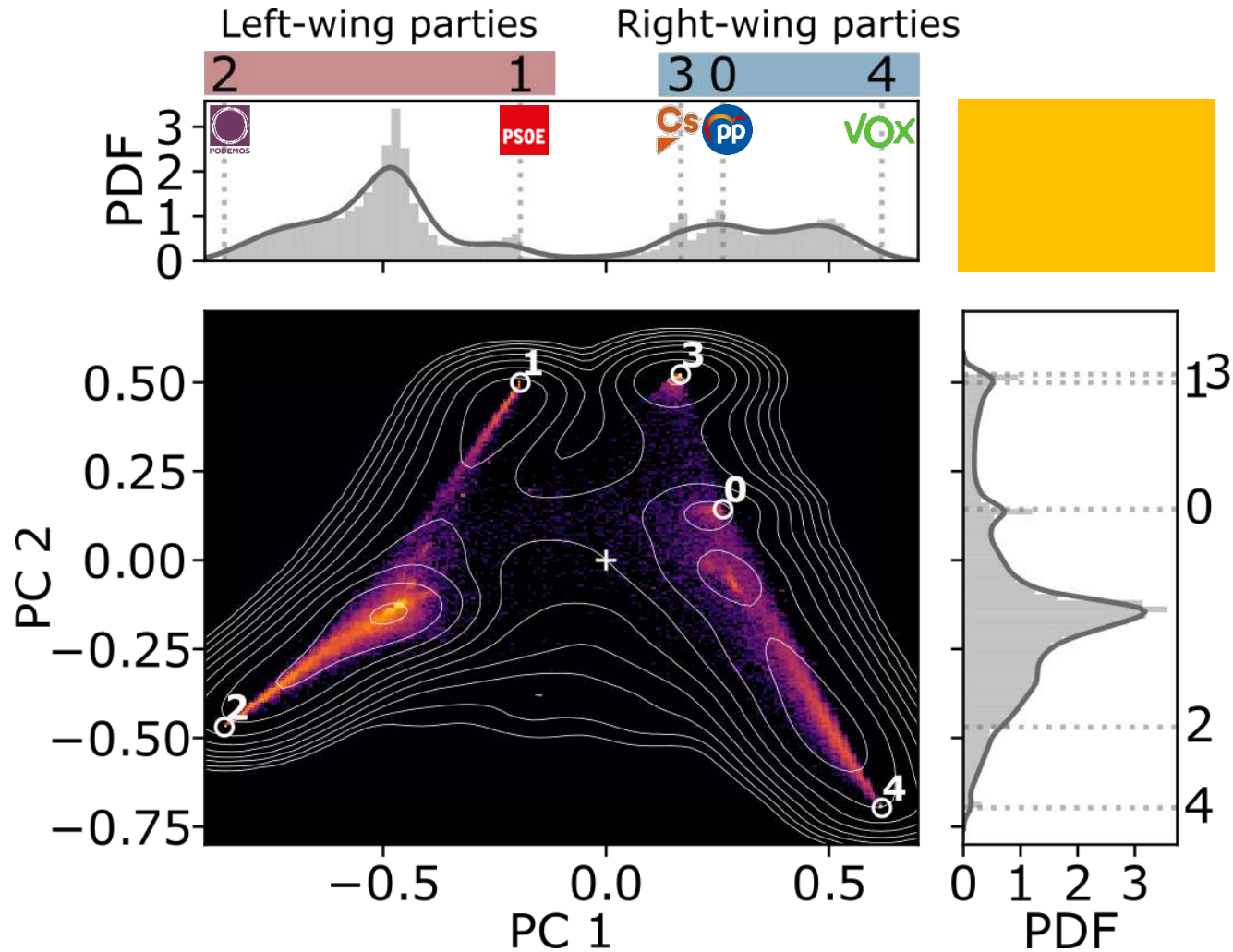


**3 - Liberalism**



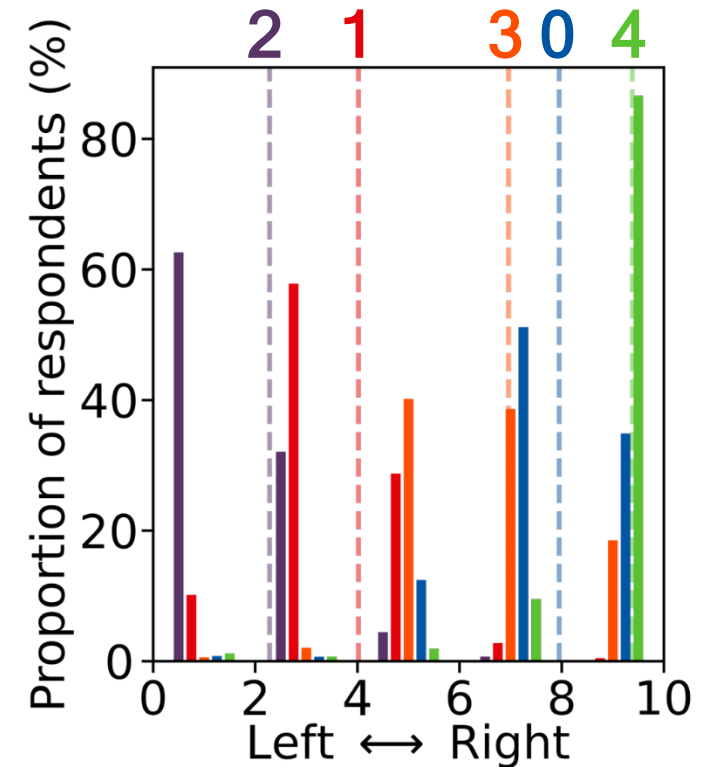
**4 - Right-wing populism**

# Pentapolar system: Spanish elections 2019



**CIS** Centro de Investigaciones Sociológicas

## National survey on parties' ideology



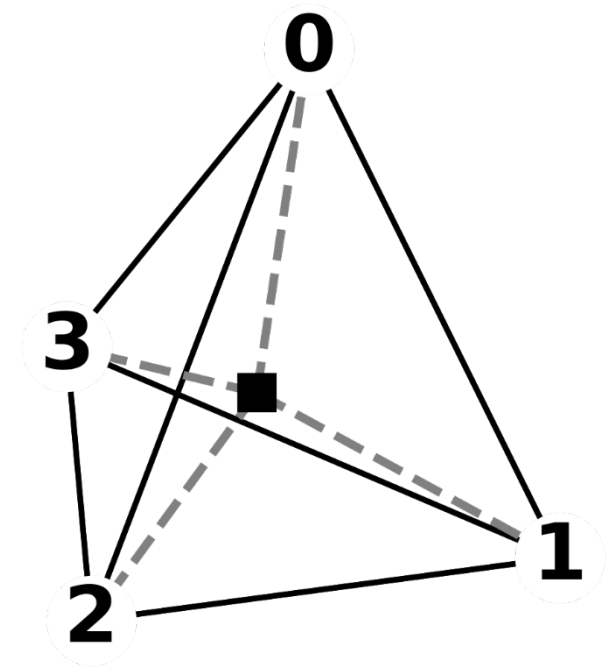
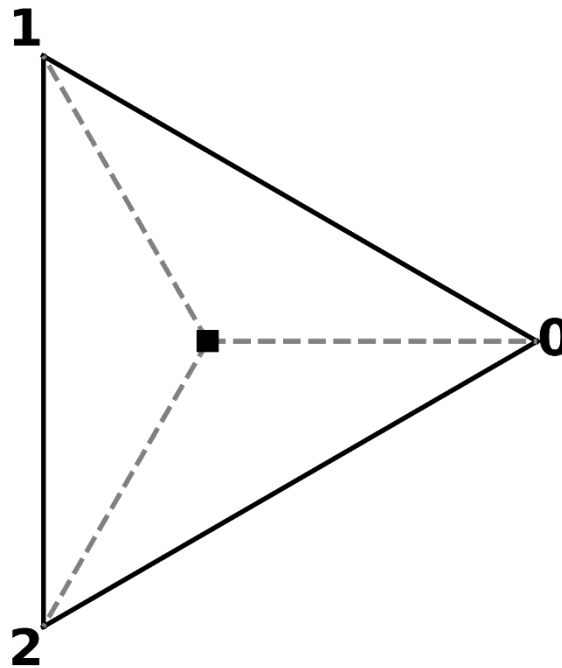
Parties' perceived extremism was computed from national-wide surveys

# Conclusions

**Bipolar  
Opinion space-1D**

**Tripolar  
Opinion space-2D**

**Quadripolar  
Opinion space-3D**

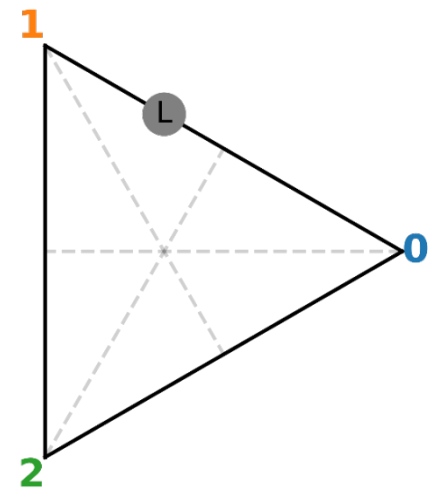
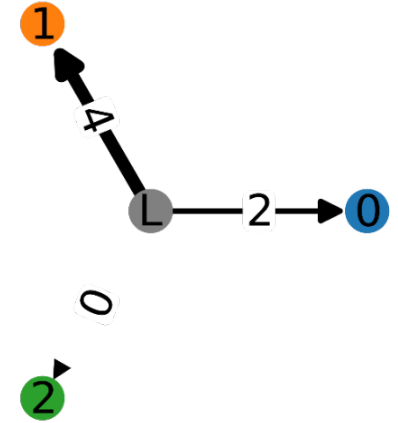


We have proposed an **unbiased model** for the ideological space of **multipolar social systems** based on the multidimensional **regular simplex**.



# Conclusions

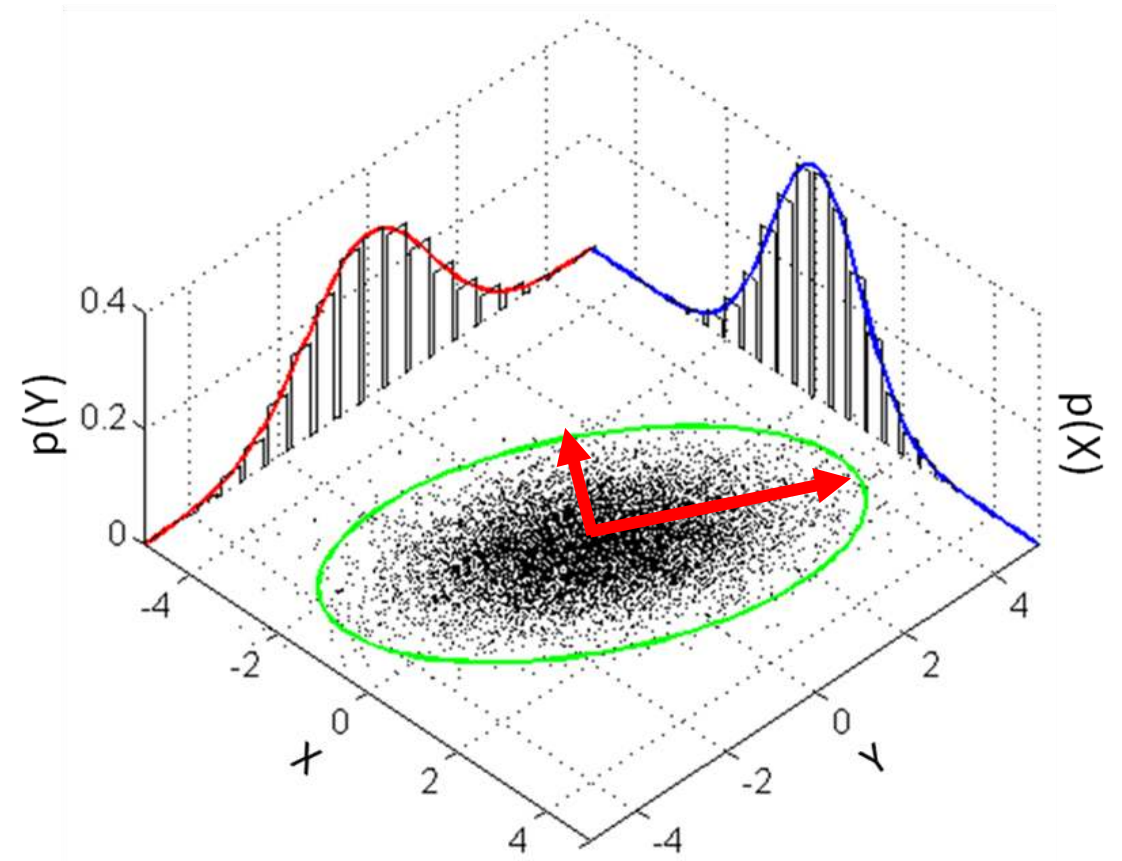
We have developed a methodology to infer **multidimensional opinion distributions** using online user interaction networks.



# Conclusions

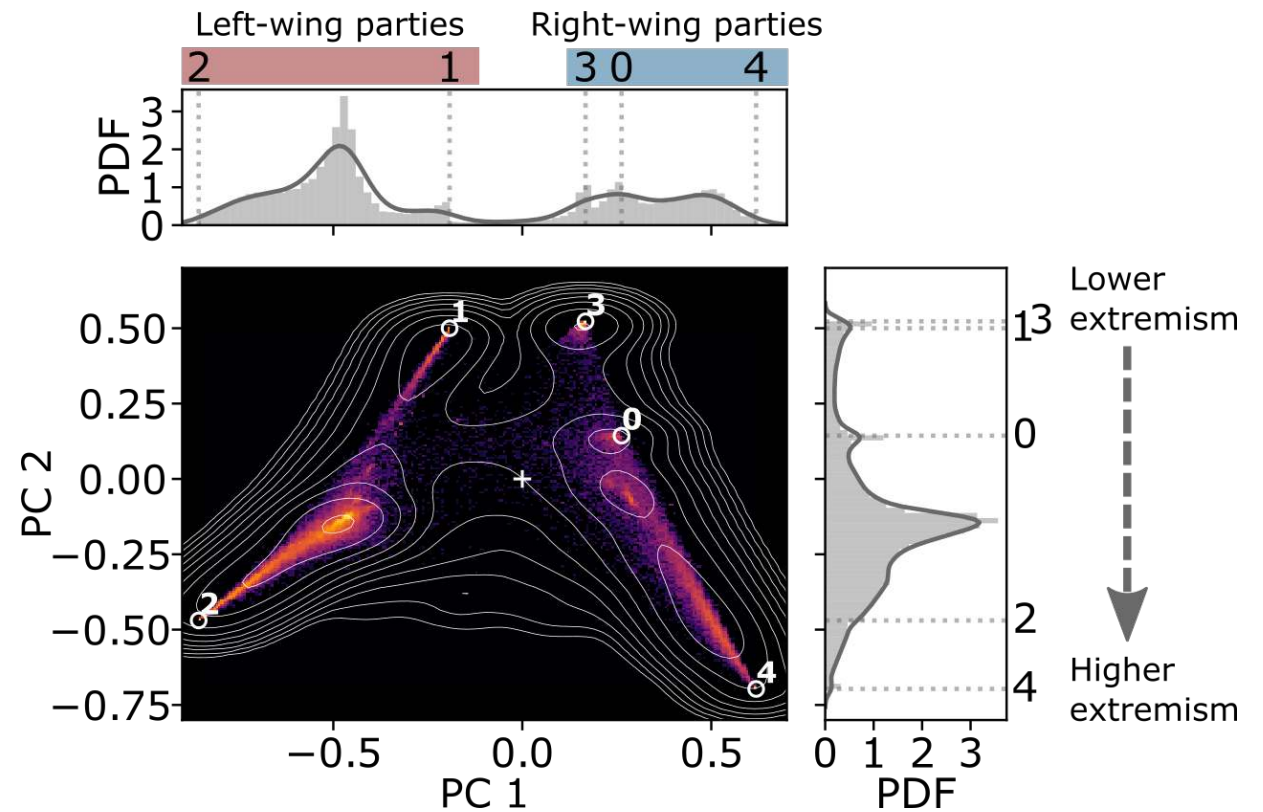
We have derived **multi-polarization** metrics from the **covariance matrix**:

- A measure of **global polarization**.
- A way to obtain the **directions of maximum polarization**.



# Conclusions

We have **quantitatively validated** the traditional left / right scheme of party systems and revealed new **non-trivial axes of polarization** in real-world multipolar contexts.



# Collaborators



Julia Atienza-Barthelemy



Samuel Martin-Gutierrez



Juan Carlos Losada



Juan Pablo Cárdenas



Alfredo J Morales



Javier Borondo



Gastón Olivares Fernández



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# Thanks for your attention!

[rosamaria.benito@upm.es](mailto:rosamaria.benito@upm.es)

[www.gsc.upm.es/gsc3](http://www.gsc.upm.es/gsc3)

Samuel Martin-Gutierrez, Juan C. Losada, Rosa M. Benito,  
*Multipolar social systems: Measuring polarization beyond dichotomous contexts*  
***Chaos, Solitons and Fractals*** 169 (2023) 113244