The nonlinear economy (I): How resource constrains lead to business cycles

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Abstract

We explore the nonlinear dynamics of a macroeconomic model with resource constraints. The dynamics is derived from a production function that considers capital and a generalized form of energy as inputs. Energy, the new variable, is depleted during the production process and has to be renewed, whereas capital grows with production and decreases from depreciation. Dependent on time scales and energy related control parameters, we obtain steady states of high or low production, but also sustained oscillations that show properties of business cycles. We also find conditions for the coexistence of stable fixed points and limit cycles. Our model allows to specify investment and saving functions for Kaldor's model of business cycles. We provide evidence for an endogenous origin of business cycles if depleting resources are taken into account.

Business cycles bear similarities to self-sustained oscillations in nonlinear dynamics. The periodic occurrence of boom, recession, depression and recovery phases in economic systems is an empirical fact. But the reasons for business cycles are still debated. Are they induced by exogenous shocks, or do they result from the endogenous nonlinear coupling of economic dynamics? We support the endogenous explanation by providing a model that generates business cycles when considering a depleting resource. This depletion is reflected in a production function for economic output dependent on the input of capital and energy. Using this production function, we derive a nonlinear dynamics that allows for the coexistence of limit cycles and stationary solutions of high productivity.

Dedicated to the memory of Jason Gallas

1 Introduction

For Jason Gallas nonlinear dynamics was not just a research domain, it was a lifelong passion. His impressive list of publications covers various application areas, ranging from geophysics to cancer. Economics, however, was only touched indirectly, when studying bifurcations in competition models [17]. This is understandable. Compared to the complex dynamics of, e.g., a Belousov-Zhabotinsky Reaction, one of Jason's favorite topics [18], nonlinear economic models are rather simple, and studying macroeconomic models with the arsenal of nonlinear dynamics was fashionable during the 1980's to 2000 [30, 39, 42], not so much today.

We take the opportunity to change this perspective a bit with the hope to renew the attention of physicists and applied mathematicians for these kind of models. Our aim is not to present a completely new approach. After all, economists would not (yet?) see a need for this. In this paper, we first remind on how these macroeconomic models have been established, to then propose some extensions, and conclude by linking our approach to current research on active matter in physics.

A focus of our investigation is the resource dependence of production. The issue itself is broadly discussed, for instance as "resource dependence theory" in management science [7], but not as a model that could be formally explored. In economics, exhaustible resources play a role since Ricardo's time [13, 29]. Notably, Hotelling made important contributions to formalize the discussion [4, 8, 22]. The main approach, however, is different from ours in that it primarily deals with how the timing of resource extraction affects its value and availability over time, balancing the diminishing stock with factors such as commodity prices, wages, profit rates, and demand.

Our starting point is the neoclassical growth model where production depends on the input of capital and labor. But these are not modeled as resources that deplete during production. Instead, they continuously grow: Labor force because of population growth, capital stock because of investments. Therefore, we propose to consider a resource that is consumed during production. We use the term "energy" for it, but interpret it very broadly as a natural resource. As a consequence, output is constrained by the availability and the renewal of this resource. Growing production implies *decreasing energy*. This denotes an important difference to capital stock which is assumed to *increase* with growing production.

Our second contribution is the formal derivation of a production *dynamics*, starting from our production function. In economics, this dynamics is often studied in so-called "multiplier-accelerator" models [36, 38]. The multiplier describes the impact of an input, e.g., capital, on the expansion of production. The accelerator describes the feedback of the growing output on the input variable, e.g., the growth of capital stock through the investment of a fraction of the output. The dynamics assumes lags in this feedback process and can produce different types of steady-state solutions, including fix points, limit cycles, damped oscillations, but also unstable solutions, i.e., growing oscillations or even chaos.

Theoretical economists, physicists and applied mathematicians have particularly studied a variant of such time-delayed dynamics, the Kalecki-Kaldor model [27, 32–34, 47, 48]. One of the reasons for this interest is the complex dynamics of the model. Such a complexity is seen not as a drawback, but as an advantage, because it offers ample possibilities to generate a wealth of dynamic patterns. These are considered a precondition to explain a complex real world phenomenon: *business cycles* [16, 35, 43], originally dubbed *trade cycles*. The question whether such delays are really needed to generate the dynamics of business cycles was negatively answered already by *Baron Kaldor* himself, who wrote in 1940: "Previous attempts at constructing models of the Trade Cycle - such as Mr. Kalecki's or Professor Tinbergen's - have thus mostly been based on the assumption of statically stable situations, where equilibrium would persist if once reached; the existence of the cycle was explained as a result of the operation of certain time-lags which prevented the new equilibrium from being reached, once the old equilibrium, for some external cause, had been disturbed. In this sense all these theories may be regarded as being derived from the 'cobweb theorem'. The drawback of such explanations is that the existence of an undamped cycle can be shown only as a result of a happy coincidence, of a particular constellation of the various time-lags and parameters assumed. The introduction of the assumption of unstable positions of equilibrium at and around the replacement level provides, however, [...] an explanation for a cycle of constant amplitude irrespective of the particular values of the time-lags and parameters involved. The time-lags are only important here in determining the period of the cycle, they have no significance in explaining its existence. Moreover, with the theories of the Tinbergen-Kalecki type, the amplitude of the cycle depends on the size of the initial shock. Here the amplitude is determined by endogeneous factors and the assumption of 'initial shocks' is itself unnecessary."[24]

This long quotation sets a nice stage for our own investigations. In line with the cited research, we attempt to explain business cycles as *endogenously* created by coupled nonlinear dynamics. This contrasts other explanations of business cycles as the result of *exogenous* perturbations of an otherwise stable dynamics. But differently from the cited research, we will not utilize delayed differential equations to generate cycles, nor propose *ad hoc* nonlinear functions. Instead, we will derive the non-linear dynamics from the production function, using suitable assumptions for the dynamics of capital and energy. This will shed new light on the formal preconditions for obtaining limit cycle dynamics.

2 A macroeconomic growth model

2.1 Production function

Economic models often start from a production function

$$\hat{Y}[X_1(t), X_2(t), ...] = Y_0 + Y = Y_0 + \mathcal{F}[X_1(t), X_2(t), ...] \tag{1}$$

$\hat{Y}$ denotes the output, or production, of a macroeconomic entity. For instance, the GDP (gross domestic product) would be an output measure for a country. It is important to note that $\hat{Y}$ is the *output per time unit* Δt, e.g., one quarter or one year. So, physically speaking it is a *velocity.* Consequently, the time derivative dY/dt discussed below is analogous to an *acceleration* and the r.h.s. of the dynamics specifies economic *forces.*

The variables $X_1(t), X_2(t), ...$ denote different inputs, e.g., capital, labor, resources, and the function $\mathcal{F}[\cdot]$ describes how the input is transformed into a valuable output, similar to the alchemistic idea of transforming lead into gold. The normalization Y_0 can be seen as an equilibrium state with baseline economic activity.

Putting Eq. (1) to use requires to specify (i) the input variables X_i, (ii) their combination in a *nonlinear* function $\mathcal{F}[\cdot]$ and (iii) their possible dynamics, $X_i(t)$. Let us first solve issue (ii). An additive combination of input variables $\mathcal{F} = a_1X_1 + a_2X_2 + ...$ implies that inputs can be *substituted* to some degree, i.e., a shortage of X_1 can be compensated by an increase of X_2. A multiplicative combination, $\mathcal{F} = X_1^{a_1} \cdot X_2^{a_2} \cdots$, on the other hand highlights that inputs are essential: X_1 cannot be completely substituted by X_2. The exponents a_1, a_2 denote *elasticities*, i.e., *relative changes* of output in response to relative changes of input:

$$a_i = \frac{\partial Y/Y}{\partial X_i/X_i} = \frac{X_i}{Y}\frac{\partial Y}{\partial X_i} = \frac{\partial \ln Y}{\partial \ln X_i} \tag{2}$$

The question is how much freedom one has in choosing the nonlinear function. As we will see, the functional form is quite restricted by some fundamental assumptions. For a general derivation, we refer to [2], whereas we follow the more didactic approach of [26]. Let us consider only two inputs X_1, X_2. Hence, we need to determine $Y(X_1, X_2)$. The first fundamental assumption is to consider only *homogeneous* functions with the power n known as the degree of homogeneity:

$$Y(\alpha X_1, \alpha X_2) = \alpha^n Y(X_1, X_2) \tag{3}$$

This relation plays a role when discussing so-called *returns to scale* in economics. If we would increase the two inputs by an arbitrary factor α, then the output increases by a factor α^n. Let us consider linear homogeneity, $n = 1$. Then the production function can be expressed in terms of partial derivatives by means of the Euler theorem:

$$1 \cdot Y(X_1, X_2) = X_1 \frac{\partial Y(X_1, X_2)}{\partial X_1} + X_2 \frac{\partial Y(X_1, X_2)}{\partial X_2} \tag{4}$$

The second fundamental assumption is about *diminishing* returns to scale. It means that the output still increases with increasing input. However, its impact becomes smaller and smaller. This reflects economic reality: it does not make sense to scale up production beyond certain limits because the marginal product tends to zero. Formally:

$$\frac{\partial^2 Y(X_1, X_2)}{\partial X_1^2} < 0 \; ; \; \frac{\partial^2 Y(X_1, X_2)}{\partial X_2^2} < 0 \tag{5}$$

The third fundamental assumption is about the independence of the inputs, which allows a *separation of variables*: $Y(X_1, X_2) = G(X_1)H(X_2)$

$$\begin{aligned} G(X_1)H(X_2) &= X_1 \frac{dG(X_1)}{dX_1} H(X_2) + X_2 \frac{dH(X_2)}{dX_2} G(X_1) \\ 1 &= X_1 \frac{dG(X_1)/dX_1}{G(X_1)} + X_2 \frac{dH(X_2)/dX_2}{H(X_2)} \end{aligned} \tag{6}$$

Eq. (6) can only hold if

$$X_1 \frac{dG(X_1)/dX_1}{G(X_1)} = a_1 \; ; \quad X_2 \frac{dH(X_2)/dX_2}{H(X_2)} = a_2 \tag{7}$$

$$a_1 + a_2 = 1 \tag{8}$$

Integration then leads to

$$\begin{aligned} \int \frac{dG(X_1)/dX_1}{G(X_1)} dX_1 &= \int \frac{a_1}{X_1} dX_1 = a_1 \ln X_1 + C_1 \\ \int \frac{dH(X_2)/dX_2}{H(X_2)} dX_2 &= \int \frac{a_2}{X_2} dX_2 = a_2 \ln X_2 + C_2 \end{aligned} \tag{9}$$

With the initial condition $Y(1,1) = A = e^{(C_1+C_2)}$ one eventually finds as the functional form for the production function:

$$Y(X_1, X_2) = A X_1^{a_1} X_2^{a_2} \tag{10}$$

We will use this form in the following. The pre-factor A is known as the *total factor productivity* and describes how efficient the inputs are used, e.g., by an advanced technology. In general, A accounts for effects in total output not caused by inputs, for instance the impact of good weather on agricultural output. It should be noted that today increasing production is mostly attributed to improvements in the total factor productivity [23].

2.2 Cobb-Douglas production function

We now solve issue (i), i.e., we specify the input variables. To discuss a concrete example, we refer to the *neoclassical growth model* [44–46], a standard macroeconomic model that uses capital $K(t)$ and labor $L(t)$ as inputs. These variables are combined in a so-called Cobb-Douglas production function [6, 21, 49]:

$$\hat{Y} = Y(K,L) = A\, K^{1-\alpha} L^{\alpha} \tag{11}$$

which neglects Y_0. The two exponents denote the elasticities with respect to labor and capital:

$$\alpha = \frac{\partial \ln Y}{\partial \ln L}\ ; \quad \beta = 1 - \alpha = \frac{\partial \ln Y}{\partial \ln K} \tag{12}$$

For the dynamics of the input variables the *neoclassical growth model* assumes:

$$\frac{dK}{dt} = sY - \kappa K\ ; \quad \frac{dL}{dt} = rL \tag{13}$$

The capital stock K grows via an investment $I = sY$ that is coupled to the current production, $0 < s < 1$ being the savings rate. I.e., by means of capital there is a positive feedback between the current and the future output level. κ is the depreciation rate, i.e., the value of capital stock exponentially decays if it is not maintained. For the labor force L an exponential growth is assumed, which is inspired by population dynamics. If the net growth rate $r > 0$, then births and immigration dominate and result in an exponential increase, if $r < 0$, then deaths and emigration dominate and result in an exponential decay of the population size which is equal to the available labor force.

For the output $Y[K,L]$, an instantaneous adjustment is assumed. Instead of dY/dt, the economic model only considers dK/dt and dL/dt and postulates that Y takes its new level immediately after K and L change. In physics, this is known as a *separation of time scales*. Compared to the slow change of K and L, Y changes fast, therefore it can be assumed in *quasi-stationary* equilibrium. This reduces the discussion to a comparison of the different values of Y *before* and *after* changes of K and L.

This setup has become the canonical model for the exogenous explanation of business cycles [25]. Subsequent works have introduced additional assumptions to endogenize the causes of business cycles [14, 19, 28, 37].

2.3 Comparative statics

The simplifying assumptions of the neoclassical growth model consequently generate only a very simple dynamics, namely either exponential growth or exponential decay of output. Therefore, the modeling aim is not to study the dynamics, but only the stationary state obtained for the *reduced variable* $y = Y/L$: *output per capita*. For instance, GDP per capita is an important economic indicator to compare the wealth of countries. Dividing Eq. (11) and Eq. (13) by L results, after some straightforward transformations, into the set of equations for the reduced variables

$$y = \frac{Y}{L} = Ak^{\beta} \; ; \quad \dot{k} = \frac{d}{dt}\left(\frac{K}{L}\right) = sy - (r+\kappa)k \tag{14}$$

These equations make it obvious that large immigration rates r act similar to large depreciation rates κ, drastically reducing the output per capita, y, and, hence, the capital per capita, k, available in a given country. More important from a modeling perspective, the dynamics now always reaches a stable equilibrium state, $k_\star$, for the capital per capita. From $\dot{k} = 0$ we obtain:

$$sAk_\star^{\beta} = (r+\kappa)k \; ; \quad k_\star = \left[\frac{sA}{r+\kappa}\right]^{\frac{1}{\alpha}} \tag{15}$$

A government should focus its policy design on the optimal value s_{gold} of the savings rate, i.e., the optimal split of the output in investment and consumption. $I = sY$ is the fraction of output reinvested into the growth of the capital stock and, hence, the further growth of the economy. Therefore, only the remainder of the output $C = (1-s)Y$ is left for consumption, e.g., increasing pensions by the government. Consumption per capita in the equilibrium state $k_\star$ is given as $c_\star = (1-s)y_\star = Ak_\star^{\beta} - (r+\kappa)k_\star$ and maximizing consumption means:

$$\frac{dc_\star}{dk_\star} = 0 \; ; \quad s_{\text{gold}} = (1-\alpha) \; ; \quad k_{\text{gold}} = \left[\frac{(1-\alpha)A}{r+\kappa}\right]^{\frac{1}{\alpha}} \tag{16}$$

The policy recommendation for governments, according to the neoclassical growth model, is then to increase the savings rate s if it is below s_{gold} because this will increase both the national wealth, i.e., the capital per capita, *and* the consumption. If, however, s is larger than s_{gold}, then the recommendation is to *decrease* the savings rate i.e., to lower the national wealth at the expense of increasing consumption. This is not the place to discuss the validity of such policy implications. But OECD recommendations for wealthy countries to increase governmental expenditures are fueled by such insights.

3 Coupling between production and energy

3.1 Modifications

"The greater the prestige, the greater the opposition" also applies to the neoclassical growth model. Instead of reviewing the many criticisms and the various economic debates that followed, we will concentrate on some nonlinear dynamic aspects. To introduce these, we replace labor force as the relevant input variable. There were certainly periods in history where economic growth was predominantly driven by an exponential increase of the *population*. But nowadays this growth does not easily translate into exponential growth of *labor force*. A refined dynamics of $L(t)$, Eq. (13) should also reflect labor related issues such as unemployment, unskilled labor and working poor, lack of specialized workforce, etc. which are not further discussed here [14].

Instead, we consider, in addition to capital, $K(t)$, a different input variable, *energy*, $E(t)$. It is a general form of "energy" to reflect also other material resources needed for production. Considering resources that are *depleted* denotes a conceptual change. The production function $Y[K,L]$ uses capital and labor as essential inputs, but the process of production does *not reduce* any of the inputs. Thus, K and L act as *catalysts* for the production, very similar to catalysts in chemical reactions. They are needed for the "reaction", but are not consumed during the production. The input variable E however is consumed, i.e., the initial resource is diminished.

To better understand the consequences of this modification, we remind that production Y is a *velocity*, output per time unit. However, capital K and energy E are not *flow variables* in the system dynamic sense, i.e., quantities per *time unit*, but *stock variables* that can be accumulated or depleted. Negative values of capital would indicate debt, which is possible in principle but will not be considered here. To avoid negative values, we constrain these stock variables by *floor values* $K_f \geq 0$, $E_f \geq 0$ such that the input variables remain positive in the dynamic case. Hence, our production function reads $Y[K(t),E(t)]$ and has the general form already discussed in Section 2.1:

$$\hat{Y}[K(t),E(t)] = Y_0 + Y = Y_0 + A\left[K(t) - K_f\right]^{a_K}\left[E(t) - E_f\right]^{a_E} \;;\; a_K + a_E = 1 \qquad (17)$$

Here, we have considered a baseline output $Y_0 > 0$ for the case that no additional capital or energy is used. This value shall reflect basic economic activities, which are always present, so Y_0 is not zero as in the neoclassical growth model. The aim of all economies is to reach a level of production well above Y_0, by means of capital and energy. Hence, our production function contains of two terms where the second one reflects changes in production resulting from the input of K and E. The fact that the input variable E is consumed is the precondition for *increasing* production, i.e., decreasing energy has a positive effect.

3.2 Eigendynamics and driven dynamics

To contribute to economic modeling we introduce an explicit dynamics for production. This point deserves a broader discussion. Our reference point, the neoclassical growth model, assumes only a dynamics for the input variables, to recalculate the output instantaneously at every time step. So Y simply follows the dynamics of K and L. We instead consider an explicit dynamics $d\hat{Y}/dt$ resulting from two terms, the *eigendynamics* $P[Y]$ and the *driven dynamics* $Q[K, E]$:

$$\frac{d\hat{Y}}{dt} = \frac{dY[Y, K, E]}{dt} = P[Y] + Q[K, E] \tag{18}$$

Eigendynamics refers to changes in the production that only depend on Y, but not on input of capital and energy. We already proposed that in such a case only the baseline production Y_0 should be reached. Hence, our eigendynamics $P[Y]$ has to reflect how this baseline value is established. This can be realized by different forms of saturation dynamics. To be flexible, we choose a very general ansatz:

$$P[Y] = g_1 \left[Y_0 - Y(t)\right] + g_2 Y(t) \left[Y_0 - Y(t)\right] \tag{19}$$

This is known as the mixed source model in management science. The first term solves the so-called "cold start problem", i.e., it guarantees that production can be induced initially, without pre-existing production. The second term reflects the fact that existing production has a positive, i.e., amplifying, impact on the further growth of output. Both terms saturate at a level Y_0, and the parameters g_1, g_2 determine the specific shape of the growth dynamics. They only affect how fast the baseline production is reached, but do not determine Y_0.

The driven dynamics $Q[K, E]$ reflects changes of production resulting from the dynamic input of capital and energy. For the derivation we can use Euler's Theorem, starting from Eq. (4) with $\tilde{K} = K(t) - K_f$, $\tilde{E} = E(t) - E_f$ as our input variables $X_j(t)$:

$$Q[K, E] = \sum_j \frac{dY_j}{dt} = \sum_j \frac{\partial Y}{\partial X_j} \frac{1}{\varepsilon_j} \frac{dX_j}{dt} = \frac{\partial Y}{\partial \tilde{K}} \frac{1}{\varepsilon_K} \frac{d\tilde{K}}{dt} + \frac{\partial Y}{\partial \tilde{E}} \frac{1}{\varepsilon_E} \frac{d\tilde{E}}{dt} \tag{20}$$

In Eq. (20) we have introduced two different time scales, $dt_K = \varepsilon_K dt$ and $dt_E = \varepsilon_E dt$ to allow the dynamics of $Y(t)$ to evolve on a time scale different from the dynamics of $K(t)$ and $E(t)$. $dt \equiv dt_Y$ then refers to the dynamics of $Y(t)$. The relevance of these different scales will be demonstrated below.

To complete the formal description, we need to determine $\partial Y/\partial \tilde{K}$ and $\partial Y/\partial \tilde{E}$ and provide kinetic assumptions for $d\tilde{K}/dt$ and $d\tilde{E}/dt$. In line with the use of energy as a depleting resource the dynamics of K and E can only be a *dissipative* dynamics, i.e., it has the general form:

$$\frac{dX_i(t)}{dt} = -\gamma_i \left[X_i - X_f\right] + Q_i \tag{21}$$

The damping term $-\gamma_i[X_i - X_f]$ ensures, on the one hand, that an unbounded growth is prevented and, on the other hand, that the value of the input variable stays above a *floor value* $X_f \geq 0$, i.e., remains positive. Then, the dynamic variable $X_i(t)$ converges to a stationary value, $[X_i^\star - X_f] = Q_i/\gamma_i$. The source term Q_i has to compensate dissipation, i.e., it denotes the growth or the inflow of resources. Hence, instead of a classical *conservative* system, we model a *dissipative* system. From a physical perspective the economy is a pumped system, similar to active matter, a point we will further elaborate in the discussion.

3.3 Capital input

To specify the dynamics of the input variables we start with capital as the "classic" input variable. Using the general form of the production function, Eq. (17), with $\tilde{K} = K(t) - K_f$, $\tilde{E} = E(t) - E_f$ and $a_K = 1/2$, gives

$$\frac{\partial Y}{\partial \tilde{K}} = \frac{A}{2}\tilde{K}^{-1/2}\tilde{E}^{1/2} = \frac{Y}{2\tilde{K}} \tag{22}$$

For the dynamics of capital we use the general dissipative ansatz, Eq. (21), but need to consider that it evolves at the time scale $t_K = \varepsilon_K t$:

$$\frac{1}{\varepsilon_K}\frac{d\tilde{K}}{dt} = -\kappa\left[K(t) - K_f\right] + Q_K \; ; \quad Q_K = sY(t) \; ; \quad \tilde{K} = \frac{s}{\kappa}Y \tag{23}$$

For the source term Q_K we have re-used the assumption from the neoclassical growth model, Eq. (13), i.e., a share s of the total output is invested into capital stock. The stationary solution of Eq. (23) gives us a *linear* relation between $\tilde{K}$ and Y. For the dynamics of production with reference to K we then obtain:

$$\frac{dY_K}{dt} = \frac{\partial Y}{\partial \tilde{K}}\frac{1}{\varepsilon_K}\frac{d\tilde{K}}{dt} = \left(\frac{\kappa}{2s}\right)\frac{1}{\varepsilon_K}\frac{d\tilde{K}}{dt} = \frac{\kappa}{2}Y(t) - \frac{\kappa^2}{2s}\left[K(t) - K_f\right] \tag{24}$$

3.4 Energy input

For the second input variable, energy, we obtain from the production function, Eq. (17), with $\tilde{K} = K(t) - K_f$, $\tilde{E} = E(t) - E_f$ and $a_E = 1/2$

$$\frac{\partial Y}{\partial \tilde{E}} = \frac{A}{2} \tilde{K}^{1/2} \tilde{E}^{-1/2} = \frac{Y}{2\tilde{E}} \tag{25}$$

To specify the dynamics for $\tilde{E}(t)$ at the time scale $t_E = \varepsilon_E t$, we start from a power series ansatz [40]:

$$\frac{d\tilde{E}}{dt_E} = Q_E + \tilde{E} \sum_{n=0}^{m} d_n Y^n = Q_E + d_0 \tilde{E} + d_1 \tilde{E} Y + d_2 \tilde{E} Y^2 ... \tag{26}$$

which generalizes the dissipative dynamics, Eq. (21). For Q_E we consider that energy is provided at a constant rate q, like sun radiation or a steady supply of fossil fuels. The term without Y has to reflect the dissipation, i.e., $d_0 = -c$, where c is the dissipation rate. If we restrict the power series to $n = 2$, the two remaining terms just describe a saturation dynamics if $d_2 = -\zeta$ is negative:

$$d_1 \tilde{E} Y - \zeta \tilde{E} Y^2 = \zeta \tilde{E}\; Y \left[Y_s - Y \right] \;; \quad Y_s = \frac{d_1}{\zeta} \tag{27}$$

This is the same saturation dynamics as assumed for second term in the mixed source model. It results:

$$\frac{1}{\varepsilon_E} \frac{d\tilde{E}}{dt} = q - c\tilde{E} + \zeta \tilde{E} \left[Y_s Y - Y^2 \right] \tag{28}$$

It makes sense to combine the two negative terms in Eq. (28) in a generalized dissipation term $-\gamma_E E$ with

$$\gamma_E = c + \zeta Y(t)^2 \tag{29}$$

In addition to the exponential decay of energy at a rate c, the dissipation function γ_E reflects that *any* change of production, *positive or negative*, requires to consume additional energy [11, 12]. Therefore, it is not a constant, but depends on the *squared* change of production, $Y(t)^2$. The parameter ζ captures the inefficiency in using the resource E to boost production. The higher ζ, the more the energy is reduced to change production by a given amount.

For $Y(t) \to 0$, i.e., no production, we obtain from Eq. (28) in the stationary limit the source value for energy, $E_Q - E_f = q/c$. For $Y(t) \to Y_0$, instead, we regain our baseline production $Y_0[K_0, E_0]$

with:

$$\left[E_0 - E_f\right] = \frac{q}{c - \zeta Y_0[Y_s - Y_0]} \tag{30}$$

The dynamics of production with reference to E eventually reads as:

$$\frac{dY_E}{dt} = \frac{\partial Y}{\partial \tilde{E}} \frac{1}{\varepsilon_E} \frac{d\tilde{E}}{dt} = \frac{Y}{2[E(t) - E_f]} q - Y \frac{c}{2} + Y^2 \frac{\zeta Y_s}{2} - Y^3 \frac{\zeta}{2} \tag{31}$$

3.5 Oscillations vs fixed points

Before introducing additional assumptions, we investigate the full dynamics. In particular, we have to consider the eigendynamics $P[Y]$, Eq. (18), of $\hat{Y}$ and the contributions $Q[K,E]$, Eq. (20) from the input variables. Putting our separate equations together at time scale t, we arrive at the three coupled nonlinear equations:

$$\begin{aligned} \frac{d\hat{Y}}{dt} = &= Y(t)\left[\frac{\kappa - c}{2} + \frac{q}{2[E(t) - E_f]} - g_1 + g_2 Y_0\right] + Y(t)^2 \left[\frac{\zeta Y_s}{2} - g_2\right] - Y(t)^3 \frac{\zeta}{2} \\ &- \frac{\kappa^2}{2s}\left[K(t) - K_f\right] + g_1 Y_0 \end{aligned} \tag{32}$$

$$\frac{d\tilde{E}}{dt} = \frac{dE(t)}{dt} = \varepsilon_E q - \varepsilon_E \left[E(t) - E_f\right]\left[c + \zeta Y_s Y(t) - \zeta Y(t)^2\right] \tag{33}$$

$$\frac{d\tilde{K}}{dt} = \frac{dK(t)}{dt} = \varepsilon_K s Y(t) - \varepsilon_K \kappa \left[K(t) - K_f\right] \tag{34}$$

We solve this dynamics numerically. As illustrated in Fig. 1 we find two significantly different outcomes; (i) a stationary production and (ii) sustained oscillations. Unfortunately, the stationary solution is $Y(t) \to Y_0$, i.e., after some intermediate oscillations we are back at the baseline scenario, while looking for a case with $Y(t) \gg Y_0$. This is obtained in the second scenario where we can verify that $\langle Y(t) \rangle > Y_0$. Thus, the average production is indeed above the baseline. More important, during certain time periods $Y(t)$ is much larger that Y_0, i.e., we see a *boom* phase of the economy. But this does not last long, and is followed by a steep decline of production that can even reach negative values, very similar to *business cycles*. These comprise *four phases* of *different duration*, (a) a *short* boom phase, (b) a *long* recession phase, (c) a *short* depression phase, and (d) a *long* recovery phase, after which a new cycle starts. This is indeed captured with our dynamics of production.

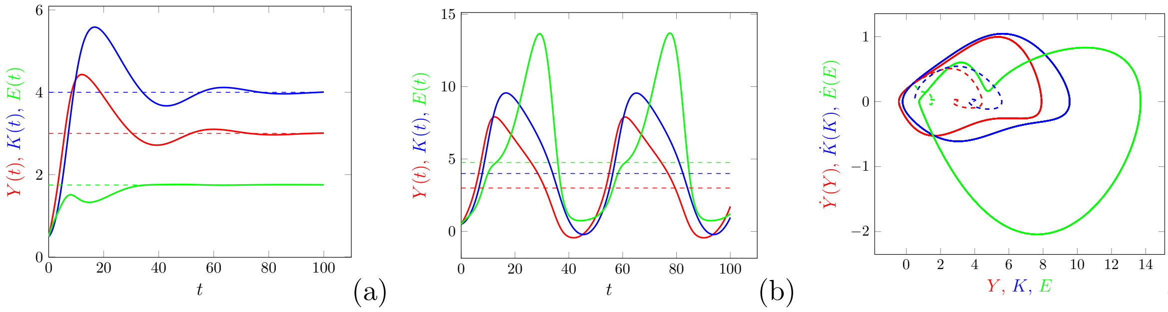

Figure 1: Production $Y(t)$ (green), capital $K(t)$ (blue) and energy $E(t)$ (green) for (a) ζ=0.04 and (b) ζ=0.02. The dashed lines give the respective baseline values Y_0=3, K_0, Eq. (23), E_0, Eq. (30). (c) Phase plots $\dot{Y}(Y)$ (red), $\dot{K}(K)$ (blue) , $\dot{E}(E)$ (green) for ζ=0.02 (solid) and ζ=0.04 (dashed). Other parameters: d_1=0.225, s=0.8, q=0.5, c=0.6, κ=0.6, Y_0=3, K_f=0, E_f=0, g_1=0.05, g_2=0.01, ε_K=0.5, ε_E=1.

4 Nonlinear oscillations and business cycles

4.1 Two-dimensional dynamics

In order to calculate a bifurcation diagram it would be convenient to reduce the full dynamics of three coupled variables to two variables. The simplest way of doing so is to assume that the dynamics of one of the input variables, K or E, relaxes very fast and therefore can be described by it's *quasistationary* equilibrium. This is equivalent of choosing $\varepsilon_K \to 0$ in Eq. (34) or $\varepsilon_E \to 0$ in Eq. (33), which results in $dK/dt = 0$ or $dE/dt = 0$. Note that this does not imply $K - K_f = 0$ or $E - E_f = 0$, instead the respective variables are tidily coupled to the production Y and therefore can still change over time via $Y(t)$. We obtain as quasi-stationary equilibria, $K_\star$, $E_\star$:

$$K_\star - K_f = \frac{s}{\kappa} Y(t)\,; \quad E_\star - E_f = \frac{q}{c - \zeta Y(t)\,[Y_s - Y(t)]} \tag{35}$$

If we choose the quasi-stationary approximation for capital, $K(t) = K_\star$, this gives the following two coupled equations for $Y(t)$ and $E(t)$:

$$\frac{d\hat{Y}}{dt} = Y(t)\left[-\frac{c}{2} + \frac{q}{2[E(t) - E_f]} - g_1 + g_2 Y_0\right] + Y(t)^2\left[\frac{\zeta Y_s}{2} - g_2\right] - Y(t)^3\frac{\zeta}{2} + g_1 Y_0 \tag{36}$$

$$\frac{d\tilde{E}}{dt} = \varepsilon_E q - \varepsilon_E\left[E(t) - E_f\right]\left[c + \zeta Y_s Y(t) - \zeta Y(t)^2\right] \tag{37}$$

This reduced dynamics still gives us the same characteristic patterns as before if we use, e.g., ε_E as control parameter: (a) for larger ε_E damped oscillations of production that converge to Y_0 over time, (b) for smaller ε_E sustained oscillations that persist over time.

If we choose instead the quasi-stationary approximation for energy, $E(t) = E_{\star}$, we have that production only depends on capital

$$\frac{d\hat{Y}[K]}{dt} = g_1 \left[Y_0 - Y(t)\right] + g_2 Y(t) \left[Y_0 - Y(t)\right] + \frac{\kappa}{2} Y(t) - \frac{\kappa^2}{2s} \left[K(t) - K_f\right] \tag{38}$$
$$\frac{d\tilde{K}}{dt} = \varepsilon_K s Y(t) - \varepsilon_K \kappa \left[K(t) - K_f\right]$$

In this case, we only obtain damped oscillations of production that converge to Y_0 over time, which is not surprising because we lack the nonlinear feedback from the energy resource.

To avoid this trivial scenario we develop a different limit case of eliminating E. We verify in Eq. (32) that the coupling between the dynamics of production and of energy is solely given by the term $Yq/(2\tilde{E})$. Using the production function, we have an expression for the energy as a function of capital:

$$\frac{\partial Y}{\partial \tilde{K}} = \frac{A}{2} \tilde{K}^{-1/2} \tilde{E}^{1/2} = \frac{\kappa}{2s} \; ; \quad \tilde{E} = \frac{\kappa^2}{A^2 s^2} \tilde{K} \tag{39}$$

Together with $Y/\tilde{K} = \kappa/s$ this can be used to replace

$$\frac{Y}{2\tilde{E}} q = \frac{A^2 s^2}{2\kappa^2} \frac{Y}{K} q = \frac{s A^2 q}{2\kappa} \tag{40}$$

We note that the parameter A^2 is related to $Y_0^2 = A^2 \tilde{K}_0 \tilde{E}_0$ with $\tilde{K}_0$ given in Eq. (23) and $\tilde{E}_0$ given in Eq. (30). Hence,

$$A^2 = \frac{Y_0 \kappa}{sq} \left(c - \zeta Y_0 [Y_s - Y_0]\right) \; ; \quad \frac{Y}{2\tilde{E}} q = Y_0 \left(\frac{c}{2} - \frac{\zeta}{2} Y_0 [Y_s - Y_0]\right) \tag{41}$$

Putting our equations together, we have a dynamics for the two coupled nonlinear equations of production and capital which is different from Eq. (38):

$$\begin{aligned} \frac{d\hat{Y}[K]}{dt} &= Y(t) \left[\frac{\kappa - c}{2} - g_1 + g_2 Y_0\right] + Y(t)^2 \left[\frac{\zeta Y_s}{2} - g_2\right] - Y(t)^3 \frac{\zeta}{2} \\ &\quad - \frac{\kappa^2}{2s} \left[K(t) - K_f\right] + Y_0 \left(g_1 + \frac{c}{2} - \frac{\zeta}{2} Y_0 [Y_s - Y_0]\right) \\ \frac{dK(t)}{dt} &= \varepsilon_K s Y(t)] - \varepsilon_K \kappa \left[K(t) - K_f\right] \end{aligned} \tag{42}$$

Precisely, the difference between these two reductions is that the quasi-stationary approximation of E results in a coupling between E and Y, Eq. (35), which gives Eq. (38), whereas Eq. (42) is based on a coupling between E and K. The latter became possible only because we used the production function as an *additional information* about the relation between E and K, coupled to Y. This dynamics becomes very promising, because we can replicate the two previous scenarios: (a) damped oscillations with $Y \to Y_0$ and (b) sustained oscillations. Additionally, it allows for a new stationary scenario: (c) convergence to a high production $Y \gg Y_0$, dependent on the choice of parameters. This will be analyzed in the following.

To make the nonlinear equation more readable we introduce the following abbreviations for the coefficients:

$$\begin{aligned} \mathrm{CL} &= \frac{\kappa - c}{2} - g_1 + g_2 Y_0 \,; \quad \mathrm{CS} = \frac{\zeta Y_s}{2} - g_2 \\ \mathrm{CQ} &= \frac{\zeta}{2} \,; \quad \mathrm{CC} = Y_0 \left(g_1 + \frac{c}{2} - \frac{\zeta}{2} Y_0 [Y_s - Y_0] \right) \end{aligned} \tag{43}$$

The cubic equation reads then in compact form:

$$\dot{Y} = \mathrm{CL}\; Y + \mathrm{CS}\; Y^2 - \mathrm{CQ}\; Y^3 + \mathrm{CC} - \frac{\kappa^2}{2s} \tilde{K}(t) \tag{44}$$

4.2 Bifurcation diagrams

The two-dimensional dynamics defined in Eq. (42) allows to calculate a bifurcation diagram from $\dot{Y} = 0$ and $\dot{K} = 0$. Because of the cubic term in the production dynamics, we can expect three stationary solutions, denoted by $Y_\star$, $K_\star$. We need to know how these solutions change if we vary control parameters of the dynamics. We have chosen ε_K because the consideration of different time scales for the dynamics of production, energy and capital is a main feature of our model. After eliminating E from the equations, ε_E plays no role. However, energy still implicitly impacts the dynamics of production via the parameters d_1 and $d_2 = -\zeta$ which define $Y_s = -d_1/d_2$, Eq. (27). The influence of energy as a depleting resource on production is a main contribution of our paper, therefore we have chosen d_1 and ζ as additional control parameters. Fig. 2 shows the bifurcation diagrams for the three different control parameters.

For our analysis we have chosen a set of parameters that yields three *fixed points*. For all three control parameters one of these fixed points is at Y_0 (equal to 1.25). Additionally, we find different regimes where either three fixed points exists or only one. The bifurcation diagram $Y_\star(\varepsilon_k)$ in Fig. 2(a) clearly shows this. It is the simplest because ε_K only affects the stability and not the values of the fixed points. For small ε_K (below 0.048), the lower and upper fixed points are unstable and the central one is a saddle point. In this case, all trajectories converge to a limit cycle, similar to the one show in Fig. 1(c) for the three-dimensional system.

Increasing the value of ε_K above 0.048, we observe for the upper fixed point a transition from unstable to stable. That means the former limit cycle now coexists with the stable fixed point

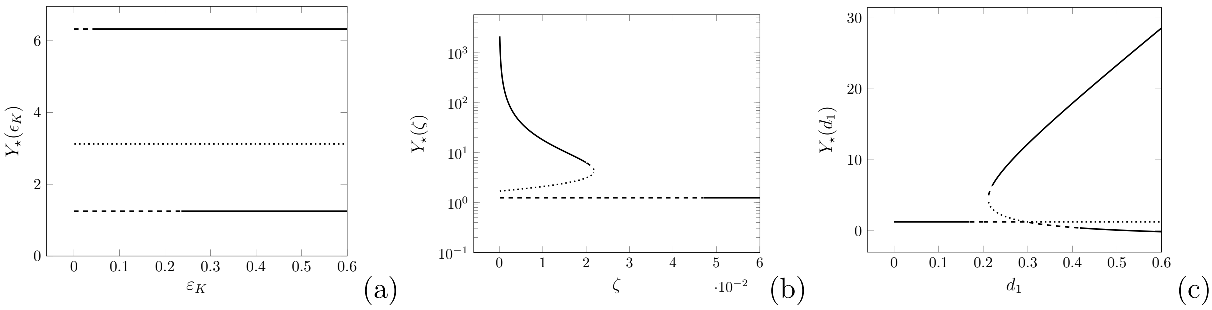

Figure 2: Bifurcation diagrams of $Y_\star$ for different control parameters: (a) ε_K, (b) $\zeta = -d_2$, (c) d_1. Dashed lines indicate unstable branches, solid lines stable branches, and dotted lines saddle points. Parameters: q_0=1, g_1=0.29, g_2=0.003, κ=0.36, s=0.8, c=0.06, d_1=0.22, d_2=0.02, Y_0=1.25, K_f=0, E_f=0, ε_K=0.06,

(for $\varepsilon_K < 0.071$). Trajectories originating close to the fixed point will converge to the fixed point, while other trajectories will converge to the limit cycle. Increasing ε_K further makes the limit cycle disappearing and all trajectories converging to the upper fixed point. For large values of ε_K (above 0.235) the lower fixed point (at Y_0) also undergoes transition from unstable to stable. Depending on the initial conditions, now trajectories will converge either to the lower or upper fixed point. The middle fixed point remains always a saddle. This means that trajectories can cross it.

The two bifurcation diagrams for the energy related control parameters d_1 and ζ are similar in that they show an inverted-s curve for $Y_\star$. To better understand this, we can treat K as a parameter. Doing so, $Y_\star$ shows three branches: the upper and lower ones are stable while the middle one unstable. This means that all trajectories will be attracted towards these two stable branches. We verify that both control parameters have a strong impact on the position of the *upper* branch that grows with d_1 and $1/\zeta$.

These three branches yield fixed points only when, in addition to $\dot{Y} = 0$, also $\dot{K} = 0$. That means K is in fact not a parameter, but a linear curve $K_\star = (s/\kappa)Y_\star$, Eq. (23), that can intersect with $Y_\star$ in up to three points, depending on the parameters. There are two *stable* points possible if $K_\star(Y_\star)$ crosses the upper or lower branches of $Y_\star$, and a *saddle point* where $K_\star(Y_\star)$ crosses the *unstable* branch of $Y_\star$ only once.

We take a closer look at Fig. 2(b), starting on the right and decreasing ζ. For large values of ζ, only one stable fixed point at Y_0 exists and all trajectories will converge to it. Decreasing ζ below 0.047, this fixed point changes from stable to unstable and the limit cycle appears. This transition resembles a Van-der-Pol oscillator. At $\zeta = 0.218$ a second fixed point appears which subsequently bifurcates into an upper unstable point and a middle saddle point. Further decreasing ζ to values below 0.02, the upper fixed point undergoes a transition from stable to unstable. The limit cycle briefly coexists with the stable fixed point, i.e., trajectories reach the limit cycle or converge to the fixed point depending on initial conditions. Slightly decreasing ζ, the limit cycle disappears and all trajectories converge to the upper fixed point whose value becomes larger with smaller ζ. Interestingly, because Y_0 remains unstable, trajectories originating below the middle fixed point first need to go below Y_0 before they can overcome the unstable point and converge to the stable one.

The bifurcation diagram for d_1, Fig. 2(c) has a similar interpretation. For low values of d_1 only a stable fixed point at Y_0 exists. Increasing d_1, we observe for this fixed point a transition from stable to unstable when the limit cycle appears, similar to a Van-der-Pol oscillator. For larger values of d_1 a second fixed point appears which then bifurcates into an unstable upper fixed point and a middle saddle point. When the upper unstable fixed point undergoes a transition to a stable point, the limit cycle first coexists with the stable point. When the limit cycle disappears, all trajectories converge to the upper fixed point. For even larger values of d_1 the middle fixed point joins the lower point at Y_0 and then bifurcates again, yielding a saddle point at Y_0 and an unstable fixed point lower than Y_0.

For $d_1 > 0.416$ the lower fixed point changes to a stable fixed point. That means, depending on initial conditions trajectories can converge either to a high value $Y \gg Y_0$ or a very low value $Y \ll Y_0$. This is the regime we were interested to find: A high *and* stable production. The fact that it coexists with a regime of low and stable production illustrates the risk for the economic system. For the same parameters the initial conditions impact whether the system ends up in a fortunate or an unfortunate regime.

As the discussion shows, the relations between the two non-trivial manifolds resulting from $\dot{Y} = 0$ and $\dot{K} = 0$ generate a complex dynamics for our production model. A particularly interesting feature is the *coexistence* of stable fixed points and cycles for certain parameter ranges. Hence, an economic system would be stable while the macroscopic dynamics is completely different. This coexistence of different stable solutions was already explored in other macroeconomic dynamics. For example, for the Kaldor model a coexistence of stable and oscillatory behavior was obtained from an interplay between noise and periodic forcing [20]. In a series of publications [3, 9, 10] Dieci *et al.* analyzed bifurcation processes that lead to the coexistence of multiple attractors, including stable equilibria and limit cycles. They demonstrated how small changes in parameters or initial conditions can result in drastically different long-term outcomes. These findings have been further extended to time-discrete systems [1].

4.3 Kaldor's model of business cycles

In his famous work, Kaldor [24] explains the emergence of business cycles from a mismatch between two different economic processes, investments $I(Y, K)$ and savings $S(Y, K)$, where the former is controlled by the capitalists and the latter by the workers. He proposed the dynamics of production as follow:

$$\frac{dY(t)}{dt} = I(Y,K) - S(Y,K) \tag{45}$$

The two functions are not specified, instead they are described by a number of qualitative arguments to justify their nonlinear dependence on capital K and production Y. We will come back on these arguments after presenting candidates for the two nonlinear functions $I(Y,K)$ and $S(Y,K)$. To derive formal expressions, we need to split the nonlinear dynamics of Eq. (44) into two parts for $I(Y,K)$ and $S(Y,K)$ such that the sum is preserved. This problem is not trivial because of underspecification. We give only three characteristic examples with the resulting functions plotted in Fig. 3.

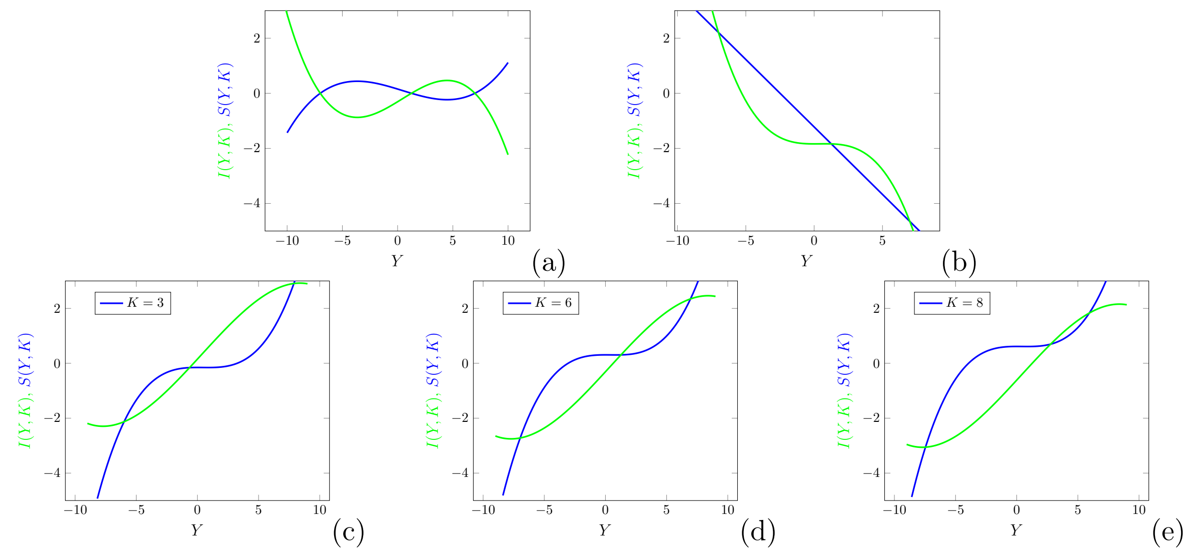

Figure 3: Investment $I(Y,K)$ and savings $S(Y,K)$ for different splits. (a) Eq. (46), (b) Eq. (47), (c)-(e) Eq. (48). Parameters: g_1=0.01, g_2=0.1, κ=0.7, s=0.8, c=0.3, d_1=0.225, d_2=0.02, Y_0=3, K=6.

A symmetric split would yield (see Fig. 3a):

$$\begin{aligned} I(Y,K) &= 0.5 \text{ CL } Y + 0.5 \text{ CS } Y^2 - 0.5 \text{ CQ } Y^3 + 0.5 \text{ CC} - 0.5\,\frac{\kappa^2}{2s}\tilde{K} \\ S(Y,K) &= -I(Y,K)\,; \text{ i.e.} \dot{Y} = 2\; I(Y,K) \end{aligned} \tag{46}$$

A linear assumption for $S(Y,K)$ leads to (see Fig. 3b):

$$\begin{aligned} I(Y,K) &= \mathrm{CS}\ Y^2 - \mathrm{CQ}\ Y^3 - \frac{\kappa^2}{2s}\tilde{K} \\ S(Y,K) &= -\mathrm{CL}\ Y - \mathrm{CC} \end{aligned} \tag{47}$$

An uneven split of the nonlinear terms between $I(Y,K)$ and $S(Y,K)$ reads instead (see Fig. 3c,d,e):

$$\begin{aligned} I(Y,K) &= \mathrm{CL}\ Y + 0.2\ \mathrm{CS}\ Y^2 - 0.25\ \mathrm{CQ}\ Y^3 + 0.5\ \mathrm{CC} - 0.5\ \frac{\kappa^2}{2s}\tilde{K} \\ I(Y,K) &= -0.8\ \mathrm{CS}\ Y^2 + 0.75\ \mathrm{CQ}\ Y^3 + 0.5\ \mathrm{CC} + 0.5\ \frac{\kappa^2}{2s}\tilde{K} \end{aligned} \tag{48}$$

All three variants lead to the same dynamics for $Y(t)$, Eq. (44). In particular all of them have the same stationary solutions. Thus, the only way to make a meaningful choice is to resort on *economic arguments*, keeping in mind that *investments* $I(Y,K)$ and *savings* $S(Y,K)$ have an economic meaning. Here, we refer to the work of Kaldor [24], but do not repeat the details of the argumentation.

First, Kaldor argues that both functions have to *monotonously increase* with production Y. This rejects Eq. (46) (Fig. 3a) because these functions are *non-monotonous* in Y. It also rejects Eq. (47) (Fig. 3b) because these functions are monotonously *decreasing* in Y. The variant of Eq. (48) (Fig. 3c,d,c) however fulfills this requirement.

Second, Kaldor distinguishes the behavior of the two functions in case of *high* and *low* capital. If capital is *low*, investments should increase with K over time to make use of many good investment opportunities. Savings, however, should decrease with K over time because of rising prices. On the other hand, if capital is *high*, investments should decrease, while savings should increase. This implies that both curves move against each other as indicated in Fig. 3(c,d). If $I(Y,K)$ moves up, $S(Y,K)$ moves down (for low levels of K) and the other way round (for high levels of K). This requirement is met by the functions of Eq. (46) and Eq. (48), but not by Eq. (47), because here $S(Y,K)$ does *not* depend on K and therefore does not move. Nevertheless, Eq. (47) would generate oscillations like the other examples.

We conclude that our proposal for the two nonlinear functions given in Eq. (48) fulfills Kaldor's requirements. It should be noted, however, that the concrete shape of our nonlinear functions depend on the chosen parameters, thus statements about the slope and the monotonous increase are restricted to this choice.

Kaldor [24] argues that two *linear* functions could still monotonously increase with Y and correctly depend on K, but they would only allow for one stationary solution. This solution would be *unstable* if the slope of $I(Y,K)$ is larger than the slope of $S(Y,K)$. Hence, the economy would either only grow or only shrink. On the other hand, if the slope of $I(Y,K)$ is smaller than the slope of $S(Y,K)$, the stationary solution would be *stable* and the economy would remain around this stable state. This, however, contradicts the broad observations of *business cycles*, which Kaldor wanted to explain with his investigations.

4.4 Relation to the Van-der-Pol oscillator

Our model generates business cycles and provides functional expressions for the dynamics. Reproducing the asymmetric duration of the phases, however, is not the merit of our efforts, it shall be attributed to the underlying general dynamics of a Van-der-Pol oscillator. This oscillator is a paragon of a non-linear system with *nonlinear* friction. To compensate this friction, oscillations require the input of energy. That means, we have a *dissipative* system and the oscillations can be classified as *limit cycles*. For a critical energy supply, we obtain a Hopf bifurcation.

The Van-der-Pol oscillator can be formalized with one second-order or with two coupled first-order differential equations. For our comparison the latter is more suitable.

$$\begin{aligned} \frac{dy(t)}{dt} &= \omega\left[y(t) - \frac{1}{3}y^3(t) - x(t)\right] \\ \frac{dx(t)}{dt} &= \frac{1}{\omega}\,y(t) \end{aligned} \tag{49}$$

Different from our two-dimensional dynamics, Eq. (42), the Van-der-Pol oscillator has only one control parameter ω that appears in both equations. To obtain oscillations $\omega > 0$ is required. Only for large ω, these oscillations remind of business cycles with short and long phases. To map the equations of the Van-der-Pol oscillator back to the Kaldor model, Eq. (45), we first have to drop the depreciation term κK in the dynamics of capital, Eq. (13), i.e., $\dot{K} = I$. If $\dot{Y} = \alpha(I - S)$ and $\dot{K} = I$ is then solved for $I(Y)$ and $K(Y)$, using Eq. (49), one finds [5]

$$\begin{aligned} I(Y) &= \frac{1}{\omega}Y \\ S(Y) &= \frac{1-\omega}{\omega}Y + \frac{Y^3}{3} + K \end{aligned} \tag{50}$$

where ω is a factor of α. Note that $S(Y,K)$ is proportional to K, similar to our Eq. (48), while $I(Y,K)$ does not depend on K, against Kaldor's requirements. While convenient for nonlinear dynamics, the fact that we only have one control parameter ω is considered a drawback for economic applications because we cannot distinguish the different processes underlying economic dynamics. Eq. (48) instead provides more flexibility.

5 Discussion

There is an ongoing debate about the origin of business cycles. Are they exogenous, triggered by external shocks such as wars, natural catastrophes, political collapse? Or, are they endogenous, that means, resulting from the internal dynamics of a country's economy, changes in consumption, inflation? The answer is probably that both endogenous and exogenous causes play a role. No surprise if *big* disruptions, like a major earthquake destroying production sites, induce a recession, even a depression of economic activities. From a modeling perspective it is more interesting how *small* disruptions, e.g., the failure of single elements, can be amplified such that a whole system collapses [31]. This requires to understand the internal feedback mechanisms that generate the system dynamics *endogenously.* In this paper, we provide a minimal model that allows to study such endogenous effects on economic growth, in particular the appearance of business cycles, in a systematic manner. Unlike previous models, which often relied on external shocks or specific nonlinear functions, our model demonstrates how these complex dynamics can arise endogenously from the interactions between production and resource constraints.

The starting point of our investigations was to propose a production function that, in addition to capital, depends on a generalized "energy": a resource that is depleted during production. Increasing production thus means decreasing energy. Our production dynamics results from this production function, together with assumptions about the dynamics of capital and energy. Here, we have chosen an ansatz that considers different time scales and a nonlinear dependence of production on energy. This way, we have derived a nonlinear dynamics of production that is able to generate business cycles endogenously, i.e., without assuming external shocks or time lags. Under the constraint of a specific coupling between capital and energy, we find additionally fixed points of the dynamics where the production is considerably higher than a baseline. While both cycles and fixed points are stable, our model is sensitive to parameter changes and initial conditions. This should be seen as a an advantage because in complex systems already small deviations can lead to instabilities, and economic systems are no exception.

The fact that energy is consumed during production drives the economy out of a thermodynamic equilibrium, which is reflected in our model. The take-up of energy, its transformation and dissipation are features of "active" matter [40]. Energy enables systems to evolve and to self-organize [15]. But in social and economic systems this does not imply a desired outcome, as witnessed by business cycles in economic activities.

As a modeling consequence of such out-of-equilibrium situation, we have to distinguish between driving and driven variables. In our case, energy is the driving variable. Its take-up and transformation increases production as the driven variable. This bears similarities to active motion, the directed movement of biological entities such as cells or animals [11]. In both cases, energy is consumed, that means *decreased* to increase production or speed. This makes our model different from related models of business cycles, which do not reflect the consumption of energy. The constrained resource sets limits to the increase of production, resulting in a saturated growth dynamics. Our model considers, in addition to the driven dynamics, also the *eigendynamics* of the system. That means, it captures the production dynamics in the *absence* of additional input, a feature not addressed in simpler macroeconomic models. It is in fact the interplay between eigendynamics and driven dynamics that generates non-trivial stationary solutions, such as high stable production or limit cycles. The coexistence of stable fixed points and limit cycles, already observed in other models of business cycles, is particularly interesting. It allows to discuss intervention mechanisms that can not only stabilize economic dynamics, but also drive it to preferred states [41], offering a new perspective on business cycle dynamics.

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