Framework for cascade size calculations on random networks

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We present a framework to calculate the cascade size evolution for a large class of cascade models on random network ensembles in the limit of infinite network size. Our method is exact and applies to network ensembles with almost arbitrary degree distribution, degree-degree correlations, and, in case of threshold models, for arbitrary threshold distribution. With our approach, we shift the perspective from the known branching process approximations to the iterative update of suitable probability distributions. Such distributions are key to capture cascade dynamics that involve possibly continuous quantities and that depend on the cascade history, e.g., if load is accumulated over time. As a proof of concept, we provide two examples: (a) Constant load models that cover many of the analytically tractable casacade models, and, as a highlight, (b) a fiber bundle model that was not tractable by branching process approximations before. Our derivations cover the whole cascade dynamics, not only their steady state. This allows us to include interventions in time or further model complexity in the analysis.

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I. INTRODUCTION

Many models in statistical physics describe the dynamics of interacting particles that are characterized by a binary state, e.g. $\{0,1\}$, healthy or failed, etc. Examples include the zero-temperature random-field Ising model [1], percolation models [2], such as the k-core percolation model on random graphs [3], or threshold models [4–6]. The Bak-Tang-Wiesenfeld sandpile model [7] has even become a paradigm for the study of self-organized criticality.

The dynamics of these models can also be interpreted as a cascade process where the change in the state of one particle subsequently impacts the state of the neighbors. Such cascade models find applications in different fields, for instance, in systemic risk analyses of financial or economic systems [8–10], information propagation in form of voter models or models for meme popularity [11], or simple epidemic spreading processes [12].

We can represent these systems of interacting particles as a network (or graph) $G = (V,E)$ consisting of nodes V and links E. Each particle is represented by a node (or vertex), its interactions by links (or edges). Each node $i \in V$ is characterized by a binary (or in general discrete) state $s_i \in \{0,1\}$ that indicates, for instance, whether i is failed, infected, activated, has a positive spin, or adopted a certain opinion ($s_i = 1$). A switch of this state influences its network neighbors, i.e., the nodes it is connected with. This can trigger further state switches and thus lead to a cascade of successive state changes.

Many models of cascade phenomena study how the network topology and specific node attributes contribute to an amplification of the dynamics such that large cascades result. Usually, the cascade size is measured by the fraction ρ of nodes in one state averaged over random graph ensembles with given degree distribution. In most of the mentioned examples, ρ can be iteratively calculated in the (thermodynamic) limit of infinitely large network size with the help of a branching process approximation, also known as local tree or heterogeneous mean-field approximation [6,13,14]. In comparison with Monte Carlo simulations, these calculations usually save computational time and effort, while they further deepen the theoretical understanding of the key factors driving a cascade. Commonly, they rely on updating compositions of generating functions which correspond to discrete probability distributions, for instance, the degree distribution of a network.

However, in models where nodes interact via a heterogeneous load redistribution mechanism, this approach breaks down, especially when continuous quantities of load can be distributed that depend on the history of the cascade. This is the case, for instance, in fiber bundle models [15], which have become one of the most prevalent models to describe fracture in materials [16]. Also neural network models [17] and multilayer perceptrons or deep-learning architectures [18] involve similar load distribution mechanisms, usually on directed networks. To enable their analysis in the thermodynamic limit of infinite network size, we present an alternative view on branching process approximations and shift the perspective towards the iterative update of suitable probability distributions. Within this analytic framework, we can correctly compute the whole time evolution of the average cascade size for a large class of cascade processes. We demonstrate this with the example of a fiber bundle model [5,19], which could, to the best of our knowledge, not be tackled analytically before. Furthermore, we show how our framework simplifies the known local tree approximations in case of threshold models on weighted networks [6,20] and extend them to allow for degree-degree correlations.

While our approach works well for models with local load distributions, we cannot tackle models that require global information, for instance, load distribution along shortest paths [21], or introduce network clustering [22]. Yet, our approach is not limited to binary state dynamics. It also applies to the

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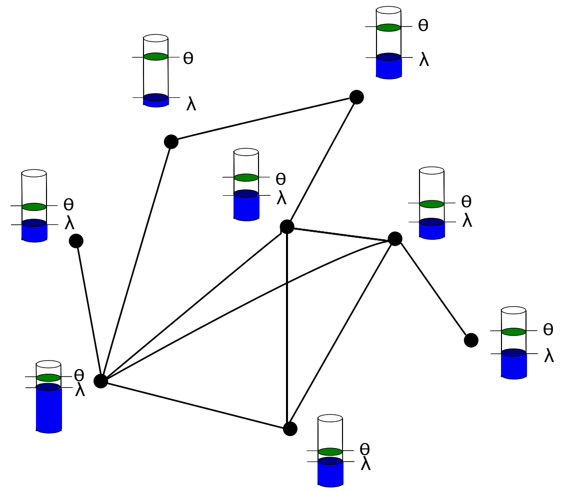

FIG. 1. Illustration of cascade model set-up. All nodes in the network carry a certain amount of load λ, which is visualized by a blue liquid in a glass. Additionally, they are equipped with a threshold θ, which is represented by a green disk. Both variables can vary between the nodes and determine a node's state s.

calculation of average node states that correspond to general discrete or continuous variables. However, in the following we stick to binary state models, as they have motivated our derivations.

II. CASCADE MODELS

The highlight of our derivations is the analytical description of the dynamics for a class of fiber bundle models. Therefore, to introduce general cascade processes, we adopt the terminology and interpretation of the fiber bundle model. Each node $i \in V$ in a network $G = (V,E)$ with node set V consisting of $N = |V|$ nodes, is associated with a fiber in a bundle to which a force is applied. A functional node i breaks or fails if it cannot withstand the force it is exposed to. Its binary state $s_i \in \{0,1\}$ indicates whether it is functional, $s_i = 0$, or failed, $s_i = 1$. The force applied to a node i is represented by a load $\lambda_i \in \mathbb{R}$. The amount that i cannot withstand anymore is given by the threshold $\theta_i \in \mathbb{R}$. Figure 1 illustrates such an initial cascade set-up. Usually, θ_i stays constant, while the load $\lambda_i(t)$ can change in the course of a cascade that evolves in discrete time steps $0,\ldots,T$. When a node's load exceeds its threshold, $\lambda_i(t) \geqslant \theta_i$, i fails. So, we have $s_i(t) = H[\lambda_i(t) - \theta_i]$, where $H(\cdot)$ denotes the Heaviside function.

The failure of a node impacts its network neighbors that are defined by the link set E of the network. For instance, in a fiber bundle, the neighboring functional fibers have to take over the load that was carried by their failed neighbors. So, when i fails at time t and it is connected with the still functional node j by a link $(i,j) \in E$, i distributes a load $l_{ij}(t)$ to j. The internal node dynamics are also visualized in Fig. 2. With the notation that $l_{ij}(t_+) = l_{ij}(t)$ for all later times $t_+ > t$, we can express the load of a node j as

$$\lambda_j(t+1) = \sum_{i=1}^{N} l_{ij}(t)s_i(t) + \lambda_j(0),$$

where $l_{ij}(t) = 0$ if there is no link $(i,j) \in E$. Thus, the failure of a node can cause subsequent failures and trigger a cascade, which ends when no thresholds are exceeded anymore. In each time step, more than one node can fail, yet, at least one node needs to fail to keep the cascade ongoing. So, the dynamics end after maximally $T = N$ steps.

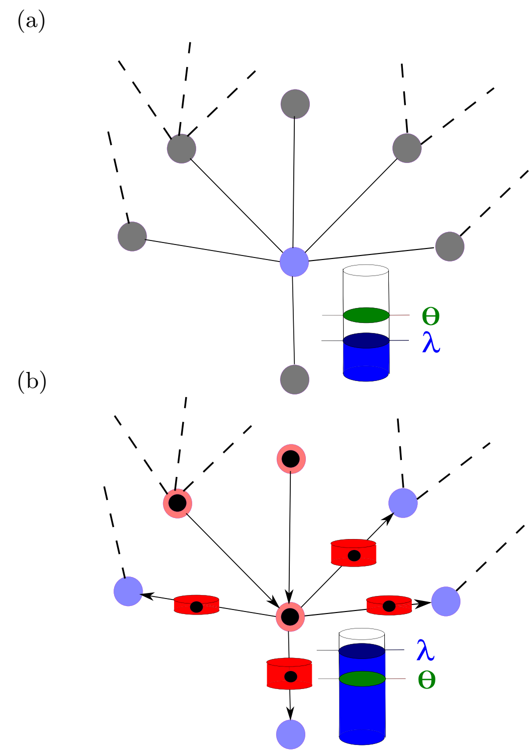

FIG. 2. Illustration of internal node dynamics. (a) A node, here colored in plain blue, is functional ($s = 0$) when $\lambda < \theta$, while (b) it fails ($s = 1$) when $\lambda \geqslant \theta$. It then distributes load, which is illustrated by red barrels with a dot, to its functional network neighbors. A failed node is colored in black with a red ring, a functional one in blue.

For simplicity, we focus on deterministic dynamics where nodes can fail only once without the option to recover. However, since our approach accurately describes the average cascade dynamics in time, recoveries and other form of interventions or changes in time can easily be integrated. Similarly, they can be extended by randomizations of failure dynamics.

Our main cascade size measure on the macro level is the fraction of failed nodes,

$$\rho_N(t) = \frac{1}{N}\sum_{i=1}^{N} s_i(t).$$

When we omit the time dependence, we usually refer to the final cascade size $\rho_N(T)$.

A. Fiber bundle model

As our main example, we study a simplified form of the fiber bundle model that has been introduced by Ref. [19] and further studied by Refs. [5,23] on regular networks. A node i that fails at time t shares its full load $\lambda_i(t)$ equally among its still functional network neighbors. Consequently, we have

$$l_{ij}(t) = A_{ij}[1 - s_j(t)]\frac{\lambda_i(t)}{k_i - n_i(t)}, \qquad (1)$$

where the adjacency matrix A has elements $A_{ij} = 1$ if $(i,j) \in E$ and $A_{ij} = 0$ otherwise. $k_i = \sum_j A_{ij}$ denotes the degree of node i and $n_i(t) = \sum_j A_{ij} s_j$ is the number of its failed network neighbors at time t.

This cascade process is Markovian given the full knowledge of all the nodes' loads $\lambda_i(t)$ and thresholds θ_i at a time t. The previous history is irrelevant for the determination of $\rho(t+1)$. However, the distributed loads $l_{ij}(t)$ are time dependent and capture the history of the cascade evolution. The process is sensitive to the order of node failures. This makes it so difficult to solve the dynamics of the cascade size $\rho_N(t)$ exactly. The key solution is to keep track of the heterogeneous distribution of $l_{ij}(t)$ *before or at time* t.

B. Constant load models

In contrast to the fiber bundle model discussed above, the load $l_{ij}(t) = w_{ij}$ distributed from a failed to a functional node is constant over time and known *a priori*. This makes these models insensitive to the precise order of failures and simplifies the local tree approximation of the corresponding cascade size. For instance, the description given by Ref. [14] of Watts model [4] has been inspired by the heterogeneous mean-field approximation of Ising models [1] and extended to configuration model type random graphs with degree-degree correlations [24,25] or multiplex topologies [10,26]. Several epidemic spreading models have been analyzed [12] in a similar context. In this paper, we show how our framework can further simplify the analytic description of such cascade dynamics. It works for the general case of weighted networks [6] and is extended to random graphs with degree-degree correlations.

C. General dynamics

We can generalize our approach also to cases where the failure of a node is no longer determined by a fixed threshold and can even relax the assumption that a node's state s_i is binary. This means to treat cases where $s_j(t) \in \mathbb{R}$ follows discrete dynamics of the form

$$s_j(t+1) = \mathcal{G}\left[\sum_{i=1}^{N} A_{ij}\mathcal{F}(s_i(t),s_j(t),k_i,k_j,\theta_i,\theta_j,\ldots,w_{ij},\rho_N(t))\right]. \tag{2}$$

We only require that $\mathcal{G}$ and $\mathcal{F}$ are measurable functions. $\mathcal{F}$ replaces the failure condition and can depend on the states of the failing as well as load receiving nodes, their degrees, other node attributes θ_i,θ_j or link attributes w_{ij} that are allowed to be random (but independent of each other), and even the cascade size $\rho_N(t)$ of the previous time step.

The arguments that we develop in the following section apply to this case. However, to simplify the derivation, we focus on the cascade models already introduced. Section III explains the concept in detail. A summary of the approach for the specific models is then provided in Secs. III E and III D.

III. LOCAL TREE APPROXIMATION

The cascade processes introduced in Sec. II evolve on a finite and fixed network G. Yet, we assume that such a network is drawn at random from an ensemble of networks with a given degree distribution $p(k)$ and degree-degree correlations $p(k,d)$. This random graph ensemble is known as extension of the configuration model [27,28], where a neighbor of a node with given degree k has degree d with probability $p(k,d)$ [29,30]. The original uncorrelated configuration model is a special case for the choice $p(k,d) = p(d)d/z$, where z denotes the average degree $z = \sum_k p(k)k$ in the network. Since the network ensemble is maximally random (i.e., it maximizes the entropy of distributions over networks) given the constraints $p(k)$ and $p(k,d)$, it is often assumed as null model for observed phenomena. It allows us to test the average influence of the degree distribution $p(k)$ and assortativity defined by $p(k,d)$ on processes running on top of such networks. In the thermodynamic limit of infinite network size $N \to \infty$, the clustering coefficient vanishes and the network topology becomes *locally treelike*, if the second moment of the degree distribution $p(k)$ is finite. This property is essential to calculate the cascade size as average over the random graphs, as the failures of neighbors can be treated as independent. This independence is sometimes also called *heterogeneous mean-field assumption*. It does not mean that nodes fail independently of each other. It just acknowledges that in one network realization a node in the network can be connected to an already failed node, while in another it might be connected to a functional one. The probability of being connected to one neighbor is independent from the probability to be connected to another neighbor.

All networks in the ensemble are undirected. Load can be distributed in both directions of a link-depending on which node fails earlier. However, we can introduce a direction by assigning weights w_{ij} and w_{ij} to a link that define a proportions or amounts of load that are distributed along a specific direction. Some weights can also be set to $w_{ij} = 0$ to obtain directed networks. Link weights are drawn initially independently at random from a distribution $p_{W(k_j,k_j)}(w)$ that can depend on the degrees of both involved nodes as introduced in Ref. [6].

In addition to the network structure, we assume that the nodes' thresholds are drawn initially independently at random from a distribution that is allowed to depend on the degree k of a node. We denote the cumulative distribution function of a node with degree k by $F_{\Theta(k)}(\theta)$. Accordingly, also the initial load that a node carries can be drawn randomly from a distribution $p_{\Lambda_{(k,0)}}(\lambda_0)$.

In summary, we study an ensemble of random graphs with fixed degree distribution $p(k)$, degree-degree correlations $p(k,d)$, possible random link weights following $p_{W(k_j,k_j)}(w)$, random thresholds following $F_{\Theta(k)}(\theta)$ and sometimes random initial loads. The ensemble is quenched in the sense that all quantities are fixed at the beginning of a cascade and do not evolve over time. Our goal is to calculate the cascade size evolution $\rho(t)$ as an average with respect to this ensemble in the thermodynamic limit of infinitely large networks.

A. Average cascade size as node failure probability

The fraction of failed nodes in an infinitely large network coincides with the probability that a node picked uniformly at random from all nodes in the network is failed. We call such

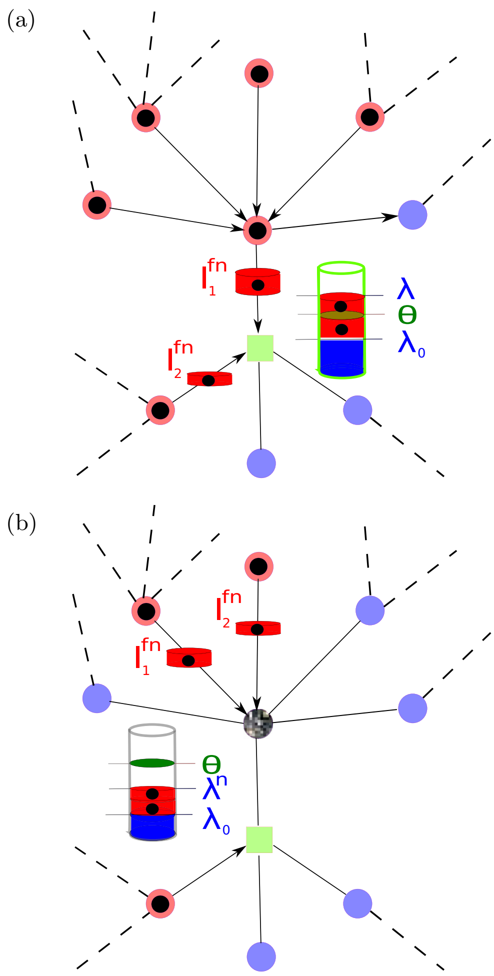

FIG. 3. Illustration of a locally treelike network structure. Dashed lines lead to other network nodes, which are allowed to be connected. (a) A focal node that is picked uniformly at random from all nodes in the network is depicted by a green square. Realizations of its load $\Lambda = \lambda$ and threshold $\Theta = \theta$ are depicted in the green glass. Two failed neighbors have distributed the load l_1^{fn} and l_2^{fn} to the focal node. (b) A focal neighbor is picked uniformly at random from the neighbors of the focal node and is colored in sandy gray. Correspondingly to (a), realizations of the load Λ^n and threshold Θ^n are shown before the failure of the neighbor.

a randomly picked node a focal node, which is visualized in green in Fig. 3(a).

After taking the limit $N \to \infty$, the average cascade size can be written as

$$\rho(t) = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} s_i^{(N)}(t)\right] =: \mathbb{P}[S(t) = 1] = \mathbb{P}[\Theta \leqslant \Lambda(t)], \quad (3)$$

where $S(t)$ denotes the state of a focal node. Its failure probability corresponds to the probability that its threshold Θ exceeds its load $\Lambda(t)$. Since these variables are random, we denote them by capital letters, while their realization is written in lowercase.

When the thresholds Θ and loads $\Lambda(t)$ are quite heterogeneous, i.e., when they are broadly distributed, mean-field approximations that only consider their average cannot lead to accurate results. Here, we present an iterative procedure to calculate their exact distributions.

To acknowledge that nodes with different degrees can have different failure probabilities, we apply the law of total probability [31] and receive

$$\rho(t) = \sum_{k=1}^{c} p(k)\mathbb{P}[S(t) = 1|K = k] = \sum_{k=1}^{c} p(k)\mathbb{P}[\Theta(k) \leqslant \Lambda(k,t)|K = k], \quad (4)$$

where $\mathbb{P}[S(t) = 1 \mid K = k]$ denotes the conditional failure probability of a node given that its degree is $K = k$. We write $\Lambda(k,t)$ and $\Theta(k)$ to indicate that the distributions of $\Lambda(t)$ and Θ can depend on the degree k of a node. The distribution of $\Theta(k)$ is defined by the initial input $F_{\Theta(k)}$. So, the goal remains to calculate the distribution of the load $\Lambda(k,t)$ that a node with degree k carries at time t.

The key in our derivation is the insight that $\Lambda(k,t)$ can be decomposed into a sum of independent random variables

$$\Lambda(k,t) = \Lambda(k,0) + \sum_{j=1}^{k} L_j^n(k,t-1) = \Lambda(k,0) + \sum_{j=1}^{N_f(k,t-1)} L_j^{\text{fn}}(k,t-1), \quad (5)$$

where $L_j^n(k,t-1)$ corresponds to the load that a neighbor has distributed to the focal node *before or at time* $t-1$. In case that a neighbor is not failed, we simply have $L_j^n(k,t-1) = 0$. $L_j^n(k,t-1)$ are independent because of the locally treelike network structure and they are equally distributed according to $p_{L^n(k,t-1)}(l)$. So, their sum is distributed as the k-manifold convolution $p^{*k}_{L^n(k,t-1)}(l)$, which can be numerically efficiently computed by a Fast Fourier Transformation [32,33]. Thus, we have reduced the problem to finding the distribution of $L_j^n(k,t-1)$.

For the case of constant load models, this view is very convenient, as the update of the probability distribution of $L_j^n(k,t-1)$ is straight forward.

Yet, when we have to keep track of the number $N_f(k,t-1)$ of failed neighbors of a node with degree k (and their time of failure), we employ an alternative view on $\Lambda(k,t-1)$. The received load can be expressed as sum over loads $L_j^{\text{fn}}(k,t-1)$ distributed by $N_f(k,t-1)$ actually failed neighbors. There, $L_j^{\text{fn}}(k,t-1) = L_j^n(k,t-1)$; $S_j^n = 1$ is not a normalized random variable. $p_{L_j^{\text{fn}}(k,t-1)}(l)$ still corresponds to the probability that the neighbor j is failed and distributes the load l to a node with degree k. It carries the total probability mass,

$$\pi(k,t-1) = \mathbb{P}[\Theta(k) \leqslant \Lambda^n(k,t-1)] = \int p_{L_j^{\text{fn}}(k,t-1)}(l) F_{\Theta(k)}(l)\, dl, \quad (6)$$

which corresponds to the probability that a neighbor of a node with degree k fails before or at time t, i.e., the probability that the neighbor's load $\Lambda^n(k,t-1)$ exceeds its threshold $\Theta(k)$. is independent of the number of neighbors $N_f(k,t-1)$ that have failed before or at time $t-1$. We only have to respect that the focal neighbor's degree D is distributed by $p_D(d) = p(k,d)$ instead of $p(k)$. Furthermore, for the evaluation of the failure condition of a focal node in Eq. (4), we are only interested in the case that the neighbor has failed before the focal node so that it can cause its failure. Consequently, only the remaining $D-1$ neighbors of the focal neighbor can have led to the failure of the focal neighbor. So, we have

$$\pi(k,t-1) = \sum_d p(k,d)\mathbb{P}\left[\Theta(d) \leqslant \Lambda(d,0) + \sum_{j=1}^{d-1} L_j^n(d,t-2)\right]. \quad (7)$$

Clearly, also the number of failed neighbors $N_f(k,t-1)$ depends on $\pi(k,t-1)$. Because of the locally treelike network structure, neighbors fail independently so that $N_f(k,t-1)$ is binomially distributed $N_f(k,t-1) \sim B[k,\pi(k,t-1)]$.

In summary, Eq. (4) can be written as

$$\rho(t) = \sum_k p(k) \sum_{f=0}^{k} \binom{k}{f} [1-\pi(k,t-1)]^{k-f} \times \int p_{\Lambda_0} * p^{*f}_{L^{\text{fn}}(k,t-1)}[l] F_{\Theta(k)}(l)\, dl, \quad (8)$$

where $p_{\Lambda_0} * p^{*f}_{L^{\text{fn}}(k,t-1)}$ is the density of $\Lambda(k,0) + \sum_{j=1}^{f} L_j^{\text{fn}}(k,t-1)$, $F_{\Theta(k)}$ the cumulative distribution function of a node's threshold, and $\pi(k,t-1)$ the failure probability of a neighbor of a node with degree k.

B. The load distributed by a failed neighbor

When the load $L_j^{\text{fn}}(k,t-1)$ that a failed neighbor distributed *before or at time* $t-1$ changes in time, we have to successively update it by the load $\Delta L_j^{\text{fn}}(k,t)$ that a neighbor failing *exactly at time* t distributes:

$$p_{L^{\text{fn}}(k,t)}(l) = p_{L^{\text{fn}}(k,t-1)}(l) + p_{\Delta L^{\text{fn}}(k,t)}(l). \quad (9)$$

For $l > 0$, we also have $p_{L^n(k,t)}(l) = p_{L^n(k,t-1)}(l) + p_{\Delta L^{\text{fn}}(k,t)}(l)$. Fiber bundle models, for instance, are challenging, because the load $\Delta L_j^n(k,t)$ that a failing neighbor distributes at time t depends on the load that this neighbor carries and its number of functional (surviving) neighbors N_s at the time of its failure: $\Delta L^{\text{fn}}(k,t) = f[\Lambda^n(t), N_s(t-1)]$.

To determine $p_{\Delta L^{\text{fn}}(k,t+1)}(l)$, we thus have to look at the situation at the time of failure of the focal neighbor, which is illustrated in Fig. 4. Let's assume the focal neighbor has degree $D = d$. Before its failure at time $t+1$, N_o of its neighbors have already failed before or at time $t-1$ without causing the failure of the focal neighbor; see also Fig. 3(b). The focal neighbor carries the load

$$\Lambda^n(d,t) = \Lambda^n(d,0) + \Lambda_o = \Lambda^n(d,0) + \sum_{j=1}^{N_o} L_j^{\text{fn}}(d,t-1),$$

which fulfills the constraint $\Lambda^n(d,t) < \Theta^n(d)$, as shown in Fig. 4(a). In the next time step t, N_n additional neighbors fail so that the focal neighbor receives the load $\sum_{i=1}^{N_n} \Delta L_j^{\text{fn}}(d,t)$. This additional load causes now the failure of the focal neighbor, as visualized in Fig. 4(a). Furthermore, N_a of the neighbors might

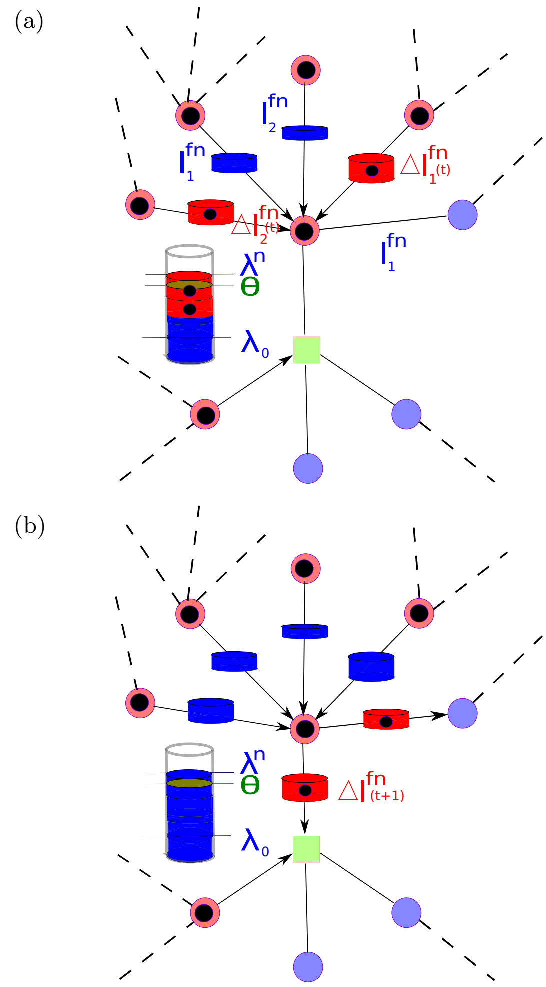

FIG. 4. Illustration of the focal neighbor at failure. (a) The plain blue load comes from neighbors that have failed before or at time $t-1$ and did not cause the failure of the focal neighbor. The additional dotted red load distributed by actually failing neighbors lead to the failure of the focal neighbor. (b) The focal neighbor distributes the load $\Delta l^{\text{fn}}(t+1)$ to the focal node in the next time step.

fail at exactly the same time as the focal neighbor so that $N_s + 1 = D - N_o - N_n - N_a$ surviving nodes receive load by the failing focal neighbor. The precise amount of load $l_{ij}(t)$ that is distributed by a failing neighbor i to a focal neighbor j is model dependent. Let's assume that it is defined by a function $f[l,n]$ that can depend on the l that the failing neighbor carries at failure and the number n of load receiving nodes or neighbors. For instance, for the fiber bundle model, f is simply $f(l,n) = l/n$. Thus, the failing focal neighbor distributes the load

$$\Delta L^{\text{fn}}(k,t+1) = f\left[\Lambda^n(D,0) + \sum_{j=1}^{N_o} L_j^{\text{fn}}(D,t-1), \times \sum_{i=1}^{N_n} \Delta L_i^{\text{fn}}(D,t), N_s + 1\right] \quad (10)$$

to the focal node, where the variables respect the constraint $\Lambda^n(D,t) < \Theta^n(D) \leqslant \Lambda^n(D,t+1)$. The variables $(N_o,N_n,N_a,N_s) \sim M[d-1, p_o(d,t), p_n(d,t), p_a(d,t), p_s(d,t)]$

follow a multinomial distribution for $D = d$. So, a single neighbor fails before or at time $t-1$ with probability $p_o(d,t) = \pi(d,t-1)$, at time t with $p_n(d,t) = \pi(d,t) - \pi(d,t-1)$, at the same time $t+1$ as the focal neighbor (with degree d) with $p_a(d,t) = \pi(d,t+1) - \pi(d,t)$, or survives $t+1$ with $p_s(d,t) = 1 - \pi(d,t+1)$.

Explicitly, this translates into the following update formula:

$$\begin{aligned} & p_{\Delta L^{\text{fn}}(k,t+1)}(l) \\ &= \sum_d p(k,d) \sum_{\underline{n}\in I_{d-1}} \frac{(d-1)!}{n_o!n_n!n_a!n_s!} p_s(d,t)^{n_s} p_a(d,t)^{n_a} \\ &\times \int_{x=0}^{\infty} p_{L(d,0)} * p^{*n_o}_{L^{\text{fn}}(d,t-1)}[f^{-1}(l,n_s+1)-x] p^{*n_n}_{\Delta L^{\text{fn}}(d,t)} \\ &\times [x]\{F_{\Theta(d)}[f^{-1}(l,n_s+1)] \\ &- F_{\Theta(d)}[f^{-1}(l,n_s+1)-x]\}\, dx, \end{aligned} \tag{11}$$

for $l > 0$, where $I_{d-1} = \{\underline{n} = (n_o,n_n,n_a,n_s) \in \{0,\dots,d-1\}^4 | n_o + n_n + n_a + n_s = d-1\}$ denotes the set of neighbor indexes with total sum $d-1$, while $f^{-1}(l,n)$ is the load that a node carries at failure after the failure of n neighbors (including the initial load) and that results in the distribution of the load l to a neighbor. While n_s neighbors survive (additionally to the focal neighbor) and n_a fail at the same time, the failing neighbor carries the total load $f^{-1}(l,n_s+1) = z + x$, where z includes the initial load and the load distributed by n_o neighbors before or at time $t-1$. This did not cause the failure of the focal neighbor yet, which implies $\theta > z$. In the present time step, n_n new neighbors fail and distribute x to the neighbor, which causes now its failure; i.e., $\theta \leqslant z + x$. The present failure of the focal neighbor thus requires $\Theta(d) \in]f^{-1}(l,n_s+1) - x, f^{-1}(l,n_s+1)]$. This event occurs with probability $\{F_{\Theta(d)}[f^{-1}(l,n_s+1)] - F_{\Theta(d)}[f^{-1}(l,n_s+1)-x]\}$. Note that we have $\sum_l p^{*n_o}_{L^{\text{fn}}(d,t-1)}[l] = p_o(d,t)^{n_o}$ and $\sum_l p^{*n_n}_{\Delta L^{\text{fn}}(d,t)}[l] = p_n(d,t)^{n_n}$. Therefore, if the response $\{F_{\Theta(d)}[f^{-1}(l,n_s+1)] - F_{\Theta(d)}[f^{-1}(l,n_s+1)-x]\} = 1$, it follows that the focal neighbor fails certainly: $\sum_l p_{\Delta L^{\text{fn}}(k,t+1)}(l) = 1$.

Equation (11) determines the iterative update of the cascade size $\rho(t)$ according to Eq. (8).

To give an overview, we summarize the algorithmic approach in more efficient form and state all necessary assumptions.

C. Summary: Local tree approximation

The time evolution of the cascade size $\rho(t)$ on (configuration model type) random graph ensembles with degree distribution $p(k)$ (with finite second moment), degree-degree correlations $p(k,d)$, initial (discrete or discretized) load distribution $p_{\Lambda(k,0)}$, load distribution function $f[l,n]$, and threshold cdf $F_{\Theta(k)}$ (which can depend on the degree k of a node) starts initially from

$$\rho(0) = \sum_k p(k) \sum_l p_{\Lambda(k,0)}(l) F_{\Theta(k)}(l),$$

$$\pi(k,0) = \sum_d p(k,d) p_{\Lambda(d,0)}(l) F_{\Theta(d)}(l)$$

$$\begin{aligned} R_n(k,f,0,l) &= p_{\Lambda(k,0)}(l) F_{\Theta(k)}(l) \text{ for all } f \\ p_{\Delta L^{\text{fn}}(k,1)}(l) &= \sum_d p(k,d) \sum_{n_s=0}^{d-1} \sum_{n_a=0}^{d-1-n_s} \binom{d-1}{n_s} \\ &\times [1-\pi(k,0)]^{n_s} \pi(k,0)^{d-1-n_s} \\ &\times R_n[d,0,0,f^{-1}(l,n_s+1)] \\ &\times p_{L^{\text{fn}}(k,1)}[l] = p_{\Delta L^{\text{fn}}(k,1)}(l) \end{aligned} \tag{12}$$

and proceeds iteratively for $t = 1,\dots,T$ as

$$\begin{aligned} R_o(k,f,t,l) &= \sum_{n_o=0}^{f} \binom{f}{n_o} R_n(k,n_o,t-1,\cdot) * p^{*(f-n_o)}_{\Delta L^{\text{fn}}(k,t)}[l] \\ R_n(k,f,t,l) &= p_{\Lambda(k,0)} * p^{*f}_{L^{\text{fn}}(k,t)}[l] F_{\Theta(k)}(l) \\ cR_n(k,f,t) &= \sum_l R_n(k,f,t,l) \\ \rho(t+1) &= \sum_k p(k) \sum_{f=0}^{k} b(k,f,t) cR_n(k,f,t) \\ \pi(k,t+1) &= \sum_d p(k,d) \sum_{f=0}^{d-1} b(d-1,f,t) cR_n(d,f,t) \\ p_a(k,t) &= \pi(k,t+1) - \pi(k,t), \\ p_s(k,t) &= 1 - \pi(k,t+1) \\ b(k,f,t+1) &= \binom{k}{f} [1-\pi(k,t+1)]^{k-f} \\ p_{\Delta L^{\text{fn}}(k,t+1)}(l) &= \sum_d p(k,d) \sum_{n_s=0}^{d-1} \sum_{\underline{n}\in I_{d-1}} \\ &\times \frac{(d-1)!}{n!n_a!n_s!} p_s(k,t)^{n_s} p_a(k,t)^{n_a} \\ &\times \{R_n[d,n,t,f^{-1}(l,n_s+1)] \\ &- R_o[d,n,t,f^{-1}(l,n_s+1)]\} \\ p_{L^{\text{fn}}(k,t+1)}[l] &= p_{L^{\text{fn}}(k,t)}[l] + p_{\Delta L^{\text{fn}}(k,t+1)}(l), \end{aligned} \tag{13}$$

for all degrees k and failed neighbors $f = 0,\dots,k$. The index $n = n_o + n_n$ in the update of $p_{\Delta L^{\text{fn}}(k,t+1)}(l)$ runs through all possible numbers of failed neighbors.

Despite the fact that the involved load distributions are discrete, we approximate the distributions on an equidistant grid in our numerical calculations. We assume that the function $f^{-1}(l,n_s+1)$ takes care of assigning the probability mass to the appropriate grid bin. The discretization allows us to compute the convolutions of load distributions by fast Fourier transformation. The previous integrals simplify thus to actual sums over discrete values of distributed load.

Next, we specify the approach for two examples, i.e., the introduced fiber bundle and constant load models.

D. Fiber bundle model

The fiber bundle model requires the knowledge of the load $\Lambda^n(k,t)$ carried by a focal neighbor of a node with degree k, as this is distributed among its $D - N_o - N_n - N_a$ functional network neighbors at the time of load distribution. Precisely, Eq. (10) specifies as

$$\Delta L^{\text{fn}}(k,t+1) = \left[\Lambda^n(D,0) + \sum_{j=1}^{N_o} L_j^{\text{fn}}(D,t-1) + \sum_{i=1}^{N_n} \Delta L_i^{\text{fn}}(D,t)\right] / [D - N_o(k,t-1) - N_n(k,t) - N_a(k,t+1)],$$

with $\Lambda^n(k,t) < \Theta^n(D) \leqslant \Lambda^n(k,t+1)$. Therefore, we have $f(l,n) = l/n$, and thus $f^{-1}(l,n_s+1) = l(n_s+1)$ in Eqs. (12) and (13).

In our numerical experiments, for simplicity, we further focus on degree uncorrelated networks, i.e., $p(k,d) = p(d)d/z$, where $z = \sum_d dp(d)$ denotes the average degree. In consequence, the load $L^{\text{fn}}(k,t+1)$ and the failure probabilities $\pi(k,t)$ become independent of the degree k.

E. Constant load models on weighted network ensembles

Constant load models are simpler to handle, as the load distributed by a neighbor with given degree d to a node with degree k does *not* depend on the cascade history. In consequence, the function $f(l,n)$ is independent of the current load l that the failing neighbor carries and of the number of functional load receiving neighbors n. The update of the load distribution $p_{L^{\text{fn}}(k,t+1)}[l]$ simplifies thus considerably and does not require us to remember the failure time of a neighbor. Instead, $f(l,n)$ is defined *a priori* by the link weights following the probability distribution $p_{W(d,k)}(w)$. The perspective of updating probability distributions still allows us to formulate the local tree approximations by Refs. [6,20] more efficiently and to extend the approach to also capture degree-degree correlations. Our starting point is Eq. (5).

Initially, all nodes with thresholds $\Theta \leqslant 0$ fail with probability $\rho(0) = \sum_k p(k)F_{\Theta(k)}(0)$ and neighbors with probability $\pi(k,0) = \sum_d p(k,d)F_{\Theta(d)}(0)$ and distribute the load,

$$p_{L^n(k,t)}(0) = [1 - \pi(k,0)] + \sum_d p(k,d)F_{\Theta(d)}(0)p_{W(d,k)}(0),$$
$$p_{L^n(k,t)}(l) = \sum_d p(k,d)F_{\Theta(d)}(0)p_{W(d,k)}(l) \text{ for } l \neq 0. \tag{14}$$

Then, each further time step $t+1$ follows from t by updating the load $L^n(k,t)$ that is distributed by a neighbor to a node with degree k. The full dynamics are captured by

$$\mathbb{P}(S^n(t) = 1|D = d) = \mathbb{P}\left[\Theta(d) \leqslant \sum_{j=1}^{d-1} L_j^n(d,t-1)\right] = \int p^{*(d-1)}_{L^n(d,t-1)}(l)F_{\Theta(d)}(l)\,dl,$$
$$\pi(k,t) = \sum_d p(k,d)\mathbb{P}(S^n = 1|D = d),$$
$$p_{L^n(k,t)}(0) = [1 - \pi(k,t)] + \sum_d p(k,d)\mathbb{P}[S^n(t) = 1|D = d]p_{W(d,k)}(0),$$
$$p_{L^n(k,t)}(l) = \sum_d p(k,d)\mathbb{P}[S^n(t) = 1|D = d]p_{W(d,k)}(l) \text{ for } l \neq 0,$$
$$\rho(t+1) = \sum_k p(k)\int p^{*k}_{L^n(k,t)}(l)F_{\Theta(k)}(l)\,dl. \tag{15}$$

We compute explicitly the failure probability $\mathbb{P}[S^n(t) = 1|D = d]$ of a focal neighbor with degree d and the failure probability $\pi(k,t)$ of a focal neighbor that is connected to a node with degree k. Then, a neighbor does not distribute any load $[L^n(k,t) = 0]$ if it is either not failed [with probability $1 - \pi(k,t)$] or is failed but the link weight is zero $[W(d,k) = 0]$. It has degree d with probability d and is additionally failed with probability $\mathbb{P}[S^n(t) = 1|D = d]$. In this case, it distributes the load l with probability $p_{W(d,k)}(l)$. This fully determines the load distribution, which allows to calculate the cascade size according to Eqs. (4) and (5).

IV. COMPARISON OF NUMERICAL LOCAL TREE APPROXIMATIONS WITH SIMULATION RESULTS

As a proof of concept, we compare the results of numerical local tree approximations (LTA) with Monte Carlo simulations on uncorrelated Poisson random graphs with degree distribution $p(k) = e^{-z}z^k/k!$ and $p(k,d) = p(d)d/z$. In principle, our derivations are exact for any degree distribution with finite second moment, for instance, power laws with finite cutoff degree. However, in practice, for large maximum degree $k_{\max} > 80$, our algorithm also requires $k_{\max}$-fold convolutions that become numerically unstable for small probability mass updates $[\pi(t+1) - \pi(t)]$. In such cases, we have to rely on approximate algorithms for the failure probabilities of high degree nodes that go beyond this manuscript.

For uncorrelated networks, the numerical iteration simplifies, as the failure probability of a neighbor $\pi(k,t)$ and the distributed load $\Delta L^{\text{fn}}(k,t)$ become independent of the degree k of the load receiving node. In this set-up, our LTA requires negligible computational resources and converges in the range of a few minutes (usually less than a minute), while simulations often require several hours.

Our simulations calculate the average cascade size over 500 independent realizations of networks consisting of $N = 10^5$

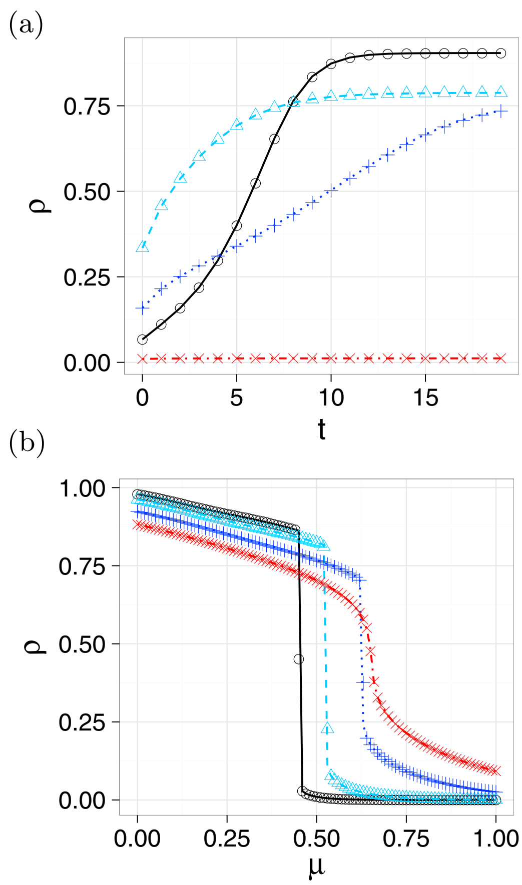

FIG. 5. Comparison of numerical local tree approximations (LTA) and simulations for the fiber bundle model and Poisson random graphs with average degree $z = 3$, where lines represent the LTA and symbols in the same color correspond to simulation results. The thresholds Θ are normally distributed with mean $\lambda_0 + \mu$ and standard deviation σ [$\Theta \sim \mathcal{N}(\lambda_0 + \mu, \sigma^2)$]. The initial load for all nodes is $\lambda_0 = 0.5$. (a) We show the cascade size evolution. Black circles belong to $(\mu,\sigma) = (0.3,0.2)$, dark blue plus signs + to $(\mu,\sigma) = (0.5,0.5)$. Light blue triangles depict $(\mu,\sigma) = (0.3,0.7)$, while red and the symbol x belong to $(\mu,\sigma) = (0.7,0.3)$. (b) Final cascade size after $T = 300$ fixed point iterations. Black circles belong to $\sigma = 0.2$ and light blue triangles to $\sigma = 0.3$. Dark blue plus signs + depict $\sigma = 0.5$, while red and the symbol x belong to $\sigma = 0.7$.

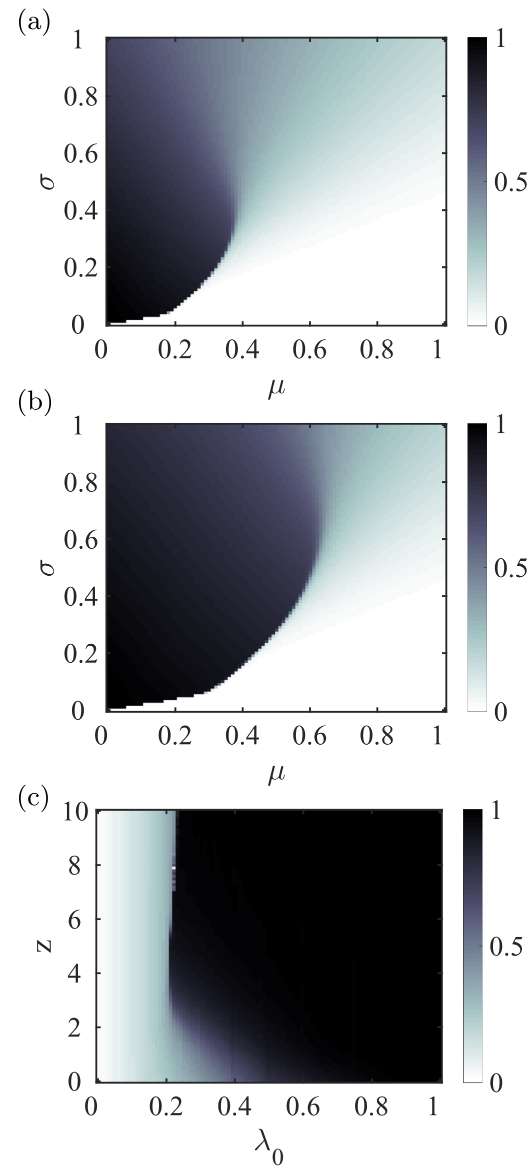

FIG. 6. Final fraction of failed nodes ρ for the fiber bundle model after 50 fixed point iterations for Poisson random graphs (with average degree z) obtained by an LTA. ρ is color coded. The darker the color, the bigger is the cascade size. Each node receives the initial load λ_0. The thresholds Θ_i are independently distributed. (a, b) Normally distributed thresholds with mean $\mu + \lambda_0$ and standard deviation σ [$\Theta \sim \mathcal{N}(\mu + \lambda_0, \sigma^2)$]. (a) $\lambda_0 = 0.3$, (b) $\lambda_0 = 0.5$. (c) Uniform threshold distribution $\Theta \sim U[0,1]$.

nodes. As in Refs. [5,23], we assume that all nodes are equipped with the same initial deterministic load λ_0 and that the thresholds follow a normal distribution independent of the degree of a node, i.e., $\Theta \sim \mathcal{N}(\mu + \lambda_0, \sigma^2)$. Correlations between degrees and thresholds are further analysed in Ref. [34].

Two constant load models have been analyzed by Ref. [6] in the described set-up. As the results of the described LTA coincide with the approach presented there, we only focus on fiber bundle models here. Figure 5 shows perfect agreement between our simulations and our LTA. The full cascade size evolution is captured as indicated by Fig. 5(a). The convergence speed to the final state and the general shape of the cascade evolution can differ substantially for different threshold parameters. Interestingly, most failures happen only after a few time steps. Figure 5(b) compares the final cascade sizes for several threshold parameters. Also the sharp regime shifts are accurately computed by our numerical calculations.

For completeness, we provide several phase diagrams for the final cascade size to give an overview of the typical influence of the model parameters. Figures 6(a) and 6(b) show normally distributed thresholds for two different values of the initial load λ_0. This extra degree of freedom, λ_0, in the model clearly influences the size of the region of big cascade sizes, but does not lead to a qualitative change of its form. As for fully connected networks [5], we observe a sharp regime shift for decreasing average threshold $\mu + \lambda_0$ and mediocre values of the standard deviation σ. Interestingly, for increasing σ the final (average) cascade size declines again for medium values of μ. We have also tested uniformly distributed thresholds, as they are often considered in fiber bundle models in the literature [15]. Figure 6(c) explores the role of the initial load λ_0 and the average degree z for uniformly distributed thresholds. For big enough initial load λ_0, almost the whole system breaks down, while ρ is negligibly small for for small enough λ_0. The average degree z influences the outcome only in a small region of λ_0.

In addition to the cascade size evolution, the LTA provides further interesting information about the time of failure of nodes conditional on their degree. This allows us to analyze, for instance, which nodes are the *main spreaders* of failures and when they fail in the course of a cascade. Interventions to reduce or enhance the cascade size can use this information to target specific nodes. In fiber bundle models, the goal is usually to prevent further failures. This can, for instance, be either achieved by saving nodes whose failure would cause many subsequent failures or by forcing them to fail early before they can accumulate a high amount of load to spread to their neighbors in case of a later failure.

Both, the failure probability of a node exactly at time t conditional on its degree k and the probability that a node fails at time t and has degree k give insights about possible intervention strategies. The conditional failure probability can be interpreted as the fraction of nodes with degree k that fail exactly at time t with respect to all nodes with degree k in the infinitely large network:

$$\mathbb{P}[S(t)=1|K=k]-\mathbb{P}[S(t-1)=1|K=k] = \mathbb{P}\left[\Theta(k)\leqslant\sum_{j=1}^{k}L_j^n(k,t-1)\right] - \mathbb{P}\left[\Theta(k)\leqslant\sum_{j=1}^{k}L_j^n(k,t-2)\right]. \tag{16}$$

The probability that a node fails at time t and has degree k is

$$\mathbb{P}[S(t)=1;K=k]-\mathbb{P}[S(t-1)=1;K=k] = p(k)\{\mathbb{P}[S(t)=1|K=k]-\mathbb{P}[S(t-1)=1|K=k]\}, \tag{17}$$

which can also be interpreted as fraction of nodes with degree k and failure time t with respect to all nodes in the network.

Both probabilities are illustrated in Fig. 7 for the studied fiber bundle model and specific parameters. Early on, the failure probability grows faster for nodes with a higher degree [Fig. 7(b)]. Initially, all nodes fail with the same probability, but already at $t=1$, when the load distribution starts, high degree nodes are stronger impacted by a cascade. Clearly, nodes with a higher degree have a higher failure risk, since more neighbors can possible fail and distribute load to them. Yet, they tend to fail early and (almost) completely. At later times, especially nodes with a smaller degree tend to fail. Since these make up for the majority of nodes in the network, their failure would need to be prevented in particular.

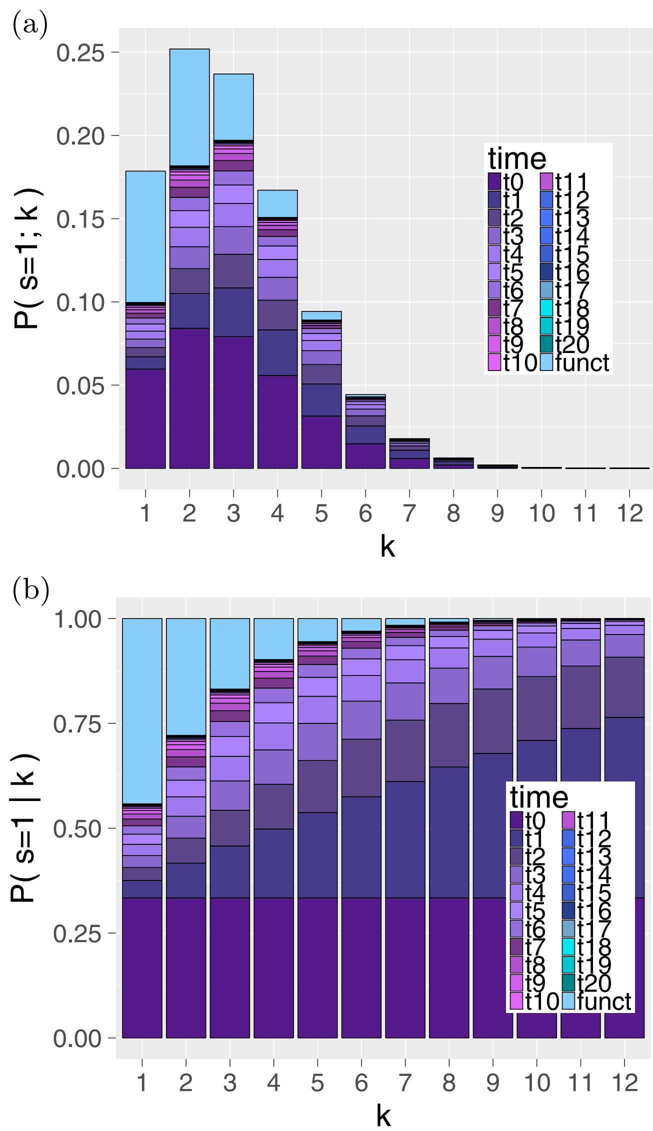

FIG. 7. Time evolution of failures for Poisson random graphs (with average degree $z=3$) obtained by an LTA for the fiber bundle model with initial load $\lambda_0=0.5$. The thresholds are independently normally distributed with mean $\mu=0.3+\lambda_0$ and standard deviation $\sigma=0.7$. The color indicates the time of failure of the respective fraction of failed nodes. (a) Probability that a node has degree k and is failed at the time indicated by the color (and position of the bar). Upper bars correspond to later failure times. (b) Probability that a node drawn uniformly at random from all nodes with degree k fails at the time indicated by the color (and position of the bar). The light blue color (top bar) corresponds to nodes that remain functional till the end of a cascade.

V. CONCLUSIONS

Our main contribution in this paper is a methodological one. We have presented a local tree approximation for specific fiber bundle models [5,19,23] on random graph ensembles with prescribed degree distribution and degree-degree correlations. Additionally, we have extended general constant load cascade models on weighted network ensembles [6] to capture degree-degree correlations. Our derivations are exact in the thermodynamic limit of infinite network size but approximate also large finite systems well.

Furthermore, we highlight that we can capture the full cascade size dynamics and not only the final cascade outcome. This is of great importance when interventions or other influencing factors shall be studied over time. Or early warning signals might be found in the growth behavior or history of the average cascade. However, it is worth noting that the dynamics in our examples span only a few time steps.

The analytic time resolution additionally allows to analyze the failure probability of nodes at specific times with respect to their degree. The role that hubs or leaves play in the amplification of cascades can be studied to inspire further strategies to prevent or enhance the growth of cascades.

In this work, we have mainly provided a proof of concept that our derivations are exact. For the presented fiber bundle and constant load models, our framework directly enables the further study of degree-degree correlations, other degree and threshold distributions, correlations between such a threshold and degree distribution and other forms of intervention.

In fact, our framework applies to a much broader category of cascade models. In comparison with known branching process approximations, we have introduced a perspective shift towards the iterative update of suitable probability distributions. These distributions belong to random variables that describe the impact of nodes on their neighborhood that have failed at any point in the past and respect the Markovian nature of the studied random processes. Several models that involve accumulation processes become analytically tractable this way. Our work bears potential to inspire the analytic investigation of even further classes of processes on networks.

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