



# Quantifying the impact of leveraging and diversification on systemic risk<sup>☆</sup>



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## ABSTRACT

The excessive increase of *leverage*, i.e. the abuse of debt financing, is considered one of the primary factors in the default of financial institutions since it amplifies potential investment losses. On the other side, portfolio *diversification* acts to mitigate these losses. Systemic risk results from correlations between individual default probabilities that cannot be considered independent. Based on the structural framework by Merton (1974), we propose a model in which these correlations arise from overlaps in banks' portfolios. Our main result is the finding of a *critical* level of diversification that separates two regimes: (i) a *safe* regime in which a properly chosen diversification strategy offsets the higher systemic risk engendered by increased leverage and (ii) a *risky* regime in which an inadequate diversification strategy and/or adverse market conditions, such as market size, market volatility and time horizon, cannot compensate the same increase in leverage. Our results are of relevance for financial regulators especially because the critical level of diversification may not coincide with the one that is individually optimal.

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## 1. Introduction

Systemic risk can be generally described as the risk that a failing agent (e.g. a bank in a financial system or a firm in a supply system) causes the failure of other agents, such that failing cascades encompass the whole system (Schweitzer et al., 2009). This approach differs from other notions of risk (Embrechts et al., 2011) which treat the default of individual agents as an extreme event for which the probability is calculated regardless of the interaction among economic agents. Instead, a systemic notion of risk requires to explicitly take into account two ingredients: (a) the stability of individual agents, and (b) the impact of interactions

among agents on their mutual stability. We apply this systemic perspective to analyse the stability of a banking system, dependent on the leverage and diversification of individual banks. The stability condition (a) is commonly expressed by the composition of its balance sheet and measured by the bank's *leverage*, e.g. by the debt-to-asset ratio. For (b), we model the interactions among agents via their common asset holdings. This represents indirect interactions via the market when agents are exposed to the same sources of risk. This approach could be extended to incorporate direct interactions via bilateral exposures (see Kiyotaki and Moore, 2002). Models with direct interactions imply cascades of defaults that propagate from a failing bank to all of its vulnerable lenders. In this sense, we provide a lower bound for systemic risk.

To quantify stability, we calculate the individual default probability using the well established structural framework by Merton (1974) in which the firm's leverage plays a major role. This approach has become popular among both academics and practitioners due to its tractability and simplicity. For a survey see (Bohn, 2000). Moreover, the Financial Stability Board (FSB, 2010) recommends it as a building block in establishing a regulatory framework that can cope with risk from systemic linkages (see e.g., Saldías, 2012).

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We study the mutual impact of failing banks by calculating their joint default probability (hereafter, “systemic default probability”) based on their individual default probabilities (as defined by the Merton framework), and their *asset correlation*. The latter increases with the level of diversification. Banks are risk-averse and hold a diversified portfolio composed of an equally weighted combination of  $n^*$  investment opportunities, also called projects. Therefore, in a market with a finite number of (uncorrelated) projects, asset correlation among banks increases with diversification due to overlapping portfolios.

In our paper, we particularly focus on the impact of *leverage* and *diversification* on the systemic default probability, and their possible interdependence. Leverage refers to the practice of investors to finance the acquisition of assets with debt in order to increase their return on equity. Clearly, the leverage practices of economic agents bear an impact on the system as a whole, however what amount of leverage can be considered “normal” or “safe” from a systemic perspective is currently unknown.<sup>1</sup> For example, in periods of booms it is natural and desirable to increase leverage, since, in upturn, real prospects for gain are improving. Conversely, such increased leverage would arguably increase systemic risk if applied in periods of market downturn. Nevertheless, there have been attempts by the [EU Parliament and EU Council \(2013\)](#) to quantify a benchmark level of leverage, based on a 5-year observation period starting from January 2013. In this paper, we do not argue for what the normal or optimal level of leverage may be. Instead, we study the effects of any increase in leverage on the systemic default probability and explore how diversification can act to oppose these effects.

The main hypothesis in our paper is that under optimal conditions diversification can compensate some of the hazards of increase in leverage. We identify two different regimes: (i) a *safe* regime in which a properly chosen diversification strategy offsets the higher systemic default probability engendered by increased leverage and (ii) a *risky* regime in which an inadequate diversification strategy and/or adverse market conditions, such as market size, market volatility and time horizon, cannot compensate the same increase in leverage.

The boundary between these two regimes is given by a critical number of projects,  $\hat{n}(f_n, f_h, N, \chi)$ , which depends on the market size  $N$ , the market conditions,  $\chi$  (which also includes the volatility and the time horizon), and two leverage levels – a lower level  $f_l$  and a higher level  $f_h$ . The safe regime is given by  $n \geq \hat{n}$ , whereas the risky regime is the opposite. We show that, especially in presence of high monitoring and transaction costs, the system can be under-diversified, i.e.,  $n^* < \hat{n}$ . Moreover, we show that  $\hat{n}$  increases with  $N$ . This last result seems to be counter intuitive, as one would expect that broader markets, in which banks can choose from a bigger pool of projects, allow for better diversification possibilities. However, larger markets reduce the effectiveness of diversification because they bear more risks which have to be compensated. Consequently, to eliminate the same proportion of risk, diversification has to increase. As a result, in larger markets, banks should diversify more to remain in the safe regime.

Next, we show that increase in leverage expands the risky regime and therefore the critical level of diversification. Interestingly, there is not always a combination of leverage and diversification to reach the safe regime. Even with increasing market size, adverse market conditions may prevent this, thus the systemic default probability is not reduced.

Our results about a critical level of diversification can be used for improving macro-prudential regulations, for example by enforcing

a ceiling to the level of leverage as recently advanced by several international bodies, see e.g. the regulation N.575/2013 of the [EU Parliament and EU Council \(2013\)](#).

### 1.1. Related literature

In addition to the literature cited above, our paper contributes to the following research lines. First, we show a different way of how the well established [Merton \(1974\)](#) model can be applied to studying joint defaults.

Second, we extend the previous literature on the optimal diversification level to a *system context*. Since the pioneering work of [Evans and Archer \(1968\)](#), the question of the optimal diversification level has been answered by determining the rate at which risk-reduction benefits are realised as the number of stocks in an equally weighted portfolio is increased. This benchmark is the so-called,  $1/n$  or *naive* rule. Previous studies consider an *unlevered* portfolio and measure the benefit of diversification with respect to the portfolio *variance* (or standard deviation) see e.g., [Johnson and Shannon \(1974\)](#), [Elton and Gruber \(1977\)](#), [Bird and Tippett \(1986\)](#), [Statman \(1987\)](#), [DeMiguel et al. \(2009\)](#), [Maillard et al. \(2010\)](#), [Pflug et al. \(2012\)](#). Differently from those studies, we extend the standard marginal volatility analysis applied to single portfolios to a *marginal (joint) default analysis* applied to multiple *levered* portfolios. Third, our work complements existing literature that recently started to investigate the role of risk diversification for the stability of the financial system ([Battiston et al., 2012](#); [Wagner, 2009](#); [Stiglitz, 2010](#); [Goldstein and Pautzner, 2004](#)). Finally, our paper is related to the literature on asset commonality (i.e., common asset holding). Indeed, asset commonality is widely considered to have been the primary vector of contagion in the recent 2007–2008 U.S. financial crisis. [Stein \(2009\)](#) identifies “crowding”, or similar portfolios among sophisticated investors, as a risk in financial markets and [Brunnermeier and Sannikov \(2010\)](#) identify portfolio overlaps as a destabilising mechanism in financial markets. According to [Allen et al. \(2012\)](#) banks swap assets to diversify their individual risk. In so doing, they increase systemic risk.

## 2. Model

### 2.1. Basic setup

We extend the [Merton \(1974\)](#) framework to a set of  $M$  banks such that for each bank  $i \in M$  the following balance-sheet identity holds true at any time  $t$

$$a_i(t) = h_i(T) + e_i(t) \quad (1)$$

where  $a_i(t)$  is the market value of bank  $i$ 's assets.  $h_i(T)$  is the promised payment of its liabilities at maturity date  $T$ .<sup>2</sup> Finally,  $e_i(t)$  is the equity value which keeps the two sides of the balance sheet even. The debt-to-asset ratio,  $h_i(T)/a_i(0) = f_i(0) = f_i \in (0, 1)$  captures the different capital structure between banks and  $1/f_i$  can be seen as an approximate measure of the credit quality of a bank.

The universe of investment opportunities is composed by the set  $\{1, \dots, N\}$  of risky projects, i.e. activities related to the real-economy such as loans to firms and households, infrastructure projects, or real-estate investments. Each of these projects has a certain price value per unit,  $v_i(t)$  at time  $t$ . As in [Merton \(1974\)](#), the

<sup>1</sup> It may even be unknowable, given the inherent complexity of financial systems.

<sup>2</sup> The equation indirectly assumes only one seniority of bonds with the same maturity date  $T$ .

price dynamics is given by a Geometric Brownian Motion of the form:

$$\frac{dv_l}{v_l(t)} = \mu_l dt + \sigma_l d\tilde{B}_l(t), \quad l = 1, \dots, N \quad (2)$$

The random shocks  $\tilde{B}_l(t)$  follow a standard Brownian motion defined on a complete filtered probability space  $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}; \mathbb{P})$ , with  $\mathcal{F}_t = \sigma\{\tilde{B}(s) : s \leq t\}$ . The drift term  $\mu_l$  describes the instantaneous risk-adjusted expected growth rate, and  $\sigma_l > 0$  is the volatility of the growth rate.

If at time  $t=0$ , bank  $i$  acquires  $x_{il}(0) \geq 0$  units of project  $l$  at a price  $v_l(0)$  with  $l = 1, \dots, N$ , its asset is given by:

$$a_i(0) := \sum_{l=1}^N x_{il}(0)v_l(0). \quad (3)$$

The investment position of bank  $i$  at  $t=0$  is denoted by the vector  $\mathbf{x}_i(0) := \{x_{i1}(0), \dots, x_{iN}(0)\}$  of units of risky projects hold. The fraction of assets invested by bank  $i$  in the project  $l$  is denoted as:

$$\omega_{il}(0) = \frac{x_{il}(0)v_l(0)}{a_i(0)}.$$

The portfolio of the bank  $i$  is the vector  $\omega_i(0) := \{\omega_{i1}(0), \dots, \omega_{iN}(0)\}$  with  $\sum_{l=1}^N \omega_{il} = 1$ . At time  $t > 0$ , the random expected return of the assets is:

$$\mathbb{E}[R_i(t)] = \omega_{i1}(t)\mu_1(t) + \dots + \omega_{iN}(t)\mu_N(t) =: \mu_i(t).$$

If at time  $t > 0$  the price has decreased compared to the initial price  $v_l(0)$ , banks face a loss, and they gain in the opposite case. The variance of the asset returns is:

$$\text{Var}[R_i(t)] = \sum_{l=1}^N \omega_{il}^2 \sigma_l^2(t) + \sum_{l=1}^N \sum_{y=1}^N \omega_{il} \omega_{iy} \sigma_{ly}(t) =: \sigma_i^2(t).$$

### 2.2. Bank optimisation problem

Bank portfolio-selection follows a *mean-variance* dominance criterion. That is, each bank  $i \in M$  selects the elements of the vector  $\{\omega_{i1}, \dots, \omega_{iN}\}$  by comparing the means and variances  $\{(\mu_1, \sigma_1), \dots, (\mu_N, \sigma_N)\}$  among projects.<sup>3</sup> However, to study the benefits of diversification in isolation, we need to consider the  $1/N$  (equally weighted) portfolio allocation<sup>4</sup>, i.e.,  $\{1/N, \dots, 1/N\}$ . This means that each bank  $i \in M$  invests the same proportion of its assets in each of the projects, i.e.  $a_i(t)/N = \text{const.}$  for all  $t \geq 0$ . This implies that Eq. (3) simplifies to  $a_i(0) = N x_{il}(0)v_l(0)$ , which holds for every project  $l$  bank  $i$  invested in.

The minimum conditions that allow us to set the  $1/N$  rule, as a benchmark, without violating the mean-variance dominance criterion, is to assume the risky projects to be indistinguishable, i.e.,  $\mu_l = \mu$  and  $\sigma_l = \sigma$  for all projects and  $\mathbb{E}(d\tilde{B}_l, d\tilde{B}_y) = 0$  for all pairs  $l, y$  in the set.<sup>5</sup> Under this condition,  $\mathbb{E}[R_i(t)] = \mu(t)$  and  $\text{Var}[R_i(t)] = \sigma^2(t)$ ,

and bank  $i$ 's utility function  $\mathbb{E}[U(R_i)]$  can be written as a smooth<sup>6</sup> function  $V(\mu, \sigma^2)$  of the mean  $\mu$  and the variance  $\sigma^2$  of  $R_i$ :  $V(\mu, \sigma^2) := \mathbb{E}[U(R_i)] = \mu - (\lambda/2)\sigma^2$ . Note that for  $\lambda > 0$  the function is strictly concave everywhere and thus exhibits risk aversion. The solution of the mean-variance maximisation problem:

$$\max_{\omega_{i1}, \dots, \omega_{iN}} \mathbb{E}[U(R_i)] = \mu - \frac{\lambda\sigma^2}{2} \quad (4)$$

subject to  $\omega_{il} \geq 0; \sum_{l=1}^N \omega_{il} = 1$ .

is:

$$\omega_i^* := \{\omega_{i1}^*, \dots, \omega_{iN}^*\} \equiv \mathbf{1}/N := \{1/N, \dots, 1/N\},$$

which is a Pareto optimal allocation.<sup>7</sup> Notice that, in a frictionless and costless market, bank  $i$  always holds the whole market portfolio i.e.,  $\omega_i^* \equiv \mathbf{1}/N$ . To prevent this unrealistic outcome, we introduce firm specific transaction costs  $c_i$  for portfolio rebalancing. In this case, bank  $i$  chooses only a subset of the available assets. Since costs are increasing in the number of projects in portfolio, diversification is extended up to the point where the marginal benefits of diversification are offset by the marginal transaction costs.<sup>8</sup>

$$\omega_i^* \equiv \mathbf{1}/\mathbf{n}_i^* := \{1/n_i^*, \dots, 1/n_i^*\} \neq \mathbf{1}/N, \quad n_i^* \leq N.$$

See Appendix A for derivation.

### 2.3. Balanced portfolios

The balance-sheet identity (1) has to be guaranteed at any time  $t$ , which implies  $a_i(t) = n_i^* x_{il}(t)v_l(t)$  for the asset side to hold true. The  $\mathbf{1}/\mathbf{n}_i^*$  rule implies that the fraction of assets allocated to each project is kept constant over time. Once  $n_i^*$  has been chosen, the bank has to keep the value  $x_{il}(t)v_l(t)$  fixed for all  $t \geq 0$ . Since  $v_l(t)$  can change for market reasons, bank  $i$  has to adjust the number of units  $x_{il}(t)$  in its portfolio with respect to the price changes. If  $v_l(t)$  goes up,  $x_{il}(t)$  should go down. In practice, at each trading date, the asset allocation is revised such that, after rebalancing, the amount invested in each of the projects is again  $a_i(t)/n_i^*$ .<sup>9</sup> That is, bank  $i$  holds the investment position  $\mathbf{x}_i(t)$  during the period  $[t - dt, t]$  and liquidates it at the prevailing prices at time  $t$ . Simultaneously bank  $i$  sets up the new portfolio  $\mathbf{x}_i(t + dt)$ . Therefore,  $\mathbf{x}_i$  represents a “contrarian” asset allocation strategy because the portfolio is rebalanced by selling “winners” and buying “losers” at any time  $t$  according to the formula:

$$x_{il}(t + dt) = \frac{1}{v_l(t + dt)} \frac{1}{n_i^*} \sum_{k=1}^N x_{ik}(t)v_k(t + dt) \quad (5)$$

for each project  $l$  in  $\mathbf{x}_i(t)$ .

Notice that, the balanced portfolio strategy is *not* a simplifying device adopted for analytical convenience. Rather, it is the result of the assumption that projects are indistinguishable. This assumption, in turn, is sufficient to allow the  $\mathbf{1}/\mathbf{n}_i^*$  rule (the standard metric used to measure the benefits of diversification) to be considered Pareto optimal.

<sup>3</sup> Under this framework, banks are rational utility maximizers and their utility function, expressed in terms of asset returns, is quadratic (see Kroll et al., 1984).

<sup>4</sup> This allocation is adopted as a metric to measure the rate at which risk-reduction benefits are realised as the number of risky projects held in the portfolio is increased. See, e.g., Johnson and Shannon (1974), Elton and Gruber (1977), Bird and Tippett (1986), Statman (1987), Maillard et al. (2010).

<sup>5</sup> Despite projects being identical in terms of means and variance, a bank's incentive to select a well diversified portfolio is maximum because all of each project's risk is due to its idiosyncratic risk and all residual variance (or systematic risk) is zero.

<sup>6</sup>  $V$  is smooth if it is a twice differentiable function of  $\mu$  and  $\sigma^2$ .

<sup>7</sup> See e.g., Rothschild and Stiglitz (1971), Samuelson (1967), Windcliff and Boyle (2004).

<sup>8</sup> Notice that, in general,  $n_i^* \neq n_j^*$  because  $c_i \neq c_j$ .

<sup>9</sup> The theory of rebalanced portfolios is developed, among others, in Kelly (1956) and Mossin (1968).

Using the expression  $a_i(t) = n_i^* x_{ii}(t)v_i(t)$ , we arrive after some transformations at the following dynamics for the asset side of bank  $i$ :

$$\frac{da_i}{a_i(t)} = \mu dt + \frac{\sigma}{\sqrt{n_i^*}} dB_i(t). \tag{6}$$

$B_i = \frac{1}{n_i^*} \sum_{l=1}^N \tilde{B}_l$  is an equally weighted linear combination of Brownian shocks s.t.  $dB_i \sim N(0, dt)$  with the condition that  $\mathbb{E}(dB_i, dB_j) = \rho_{ij}dt$ , where  $\rho_{ij}$  is the asset correlation between bank  $i$  and  $j$ .

2.4. Leverage and diversification strategy

The level of leverage is not included in the mean-variance optimisation problem of Eq. (4) because we are *not* interested in studying how the optimal number of projects in the asset side may be affected by the level of leverage. Besides, we need to consider that being based on the Merton framework, we are in a Modigliani–Miller economy where the market value of any firm is independent of its capital structure. The so-called “leverage irrelevance” principle implies that the value (and the composition, i.e., number of projects) of bank assets in Eq. (3), is not affected by the sources of financing (i.e. the leverage). That is, a bank selects the number of projects to include in its portfolio irrespective of the chosen financing mix between equity and debt. Moreover, banks are mean-variance risk averse. This means that the optimal level of diversification depends only on the distribution properties of the asset returns and namely, on their mean and variance. These moments, by construction, are unaffected by the level of leverage. Thus, leverage *cannot* and *should not* be embedded in the standard MV portfolio optimisation problem formulated in Section 2.2. With the above considerations in mind, we characterise the banks from their specific position in the leverage-diversification space, where the pair  $(f_i, n_i^*)$  defines the position of bank  $i$ . This means that each bank  $i$  has a different leverage ratio  $f_i$ , and a different optimal number  $n_i^*$  of projects in the asset side:<sup>10</sup>

$$\frac{1}{f_i} = \frac{n_i^* x_{ii}(0)v_i(0)}{h_i(T)}. \tag{7}$$

While  $n_i^*$  is the result of an optimal decision making process,  $f_i$  is used as an exogenous control parameter. Indeed, the optimal leverage policy (or the choice of the optimal financing mix) is a complex, and still open, research question that deserves to be addressed (inside the capital structure literature, in the field of corporate finance) by including, in the same modelling framework all the determinants of a firm’s optimal capital structure.<sup>11</sup> Because of these reasons, it is convenient in our setting to consider the leverage as a tuneable parameter that can take any value in its range of variation.

2.5. Asset correlation

Before moving to the study of the individual and systemic default probability in the next section, we explain the role of correlation in our setting. There exist two types of correlations – one between projects,  $\rho_{ly}$ , and another between bank assets,  $\rho_{ij}$ .

By design, we turn the correlation between projects off, i.e.,  $\rho_{ly} = 0$ . There are two main reasons for this choice. First, as

explained in Section 2.2, indistinguishable assets and no correlation between projects makes the  $1/n_i^*$  rule consistent with the mean-variance optimisation problem. The second reason is that no correlation between projects allows to isolate the effects that overlapping portfolios, resulting from higher diversification, exert on the asset correlation between banks. Were we to add in the model an arbitrary correlation matrix between projects, the impact of diversification on individual and systemic default probabilities would be confounded by both types of correlations.

We also note that the individual and systemic default probabilities estimated in Section 4 represent a *lower bound* of the real default probability because in reality the correlation of banks’ assets does not require a common portfolio<sup>12</sup>. To conclude, the correlation between bank assets equals the “overlapping correlation” matrix proposed by Tasca (2013):

$$R_{(M,M)} = \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1M} \\ \rho_{2,1} & 1 & \cdots & \rho_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{M,1} & \rho_{M,2} & \cdots & 1 \end{pmatrix} \tag{8}$$

with

$$\rho_{ij} = \frac{\sqrt{n_i^* n_j^*}}{N}.$$

3. Individual and systemic default probability

Following the structural approach proposed by Merton (1974), we assume that the default occurs if the asset value of bank  $i$  at debt maturity date  $T$  falls beneath the book value of its debt<sup>13</sup>, i.e.,  $a_i(T) \leq h_i(T)$ . More formally, given the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , the default probability  $\mathbb{P}D_i(f_i, n_i^*)$  of the bank  $i$  in the leverage-diversification space  $(f_i, n_i^*)$  is at the end of the period  $T$  defined as:

$$\begin{aligned} \mathbb{P}D_i(f_i, n_i^*) &:= \mathbb{P}[a_i(t) \leq h_i \mid a_i(0) = a_{i0}] \\ &= \mathbb{P}[\ln(a_i(t)) \leq \ln(h_i) \mid a_i(0) = a_{i0}] = \Phi_1(z_i) \end{aligned} \tag{9}$$

where  $\Phi_1(z_i)$  is the standard cumulative normal distribution function with the argument

$$z_i = -\frac{\ln(1/f_i) + \mu T - \chi/n_i^*}{\sqrt{2\chi/n_i^*}}; \quad \chi = \frac{\sigma^2 T}{2} \tag{10}$$

The constant  $\chi$  combines the effects of asset volatility and time to maturity. The probability of bankruptcy depends on the distance between the current asset value  $a_i(t)$  and the debt value  $h_i(T)$ , adjusted for the expected growth in asset value,  $\mu$ , relative to asset volatility,  $\sigma$ . We verify that the default probability is increasing in  $h_i$ , decreasing in  $a_i$  and, for  $a_i > h_i$ , increasing in  $\sigma/\sqrt{n_i^*}$ , which is in line with economic intuition. Then, increasing asset diversification (i.e., letting  $n_i^* \rightarrow N$ ) is desirable since it lowers  $\Phi_1(z_i)$ , *ceteris paribus*.

<sup>10</sup> The bank holds the selected portfolio  $1/n_i^*$  and locks the proportion of assets allocated to each project until time  $T$ .

<sup>11</sup> Furthermore, one has to consider that the choice of the optimal target leverage go beyond the pure mere control of the firm because leverage may changes according to market fluctuations and may vary with the cycles of the overall economic activities, see e.g., Adrian and Shin (2010) and Tasca and Battiston (2012).

<sup>12</sup> Because individual assets are generally correlated as they are exposed to common aggregate shocks.

<sup>13</sup> Even though in Merton (1974) framework the value of debt is marked-to-market at each date  $t < T$ , the default may occur *only* at the maturity date  $T$ , see (Merton, 1974, pp. 452–53). Therefore, it is justified to consider  $h_i(T)$  as default threshold in our model.

The individual default probability (9) refers to the default of a bank irrespective of the default of other banks.<sup>14</sup> However, in order to quantify systemic risk, we have to calculate the probability of a joint default of the whole system. In this paper, we extend the approach by Merton (1974) by calculating this probability based on the above considerations. Joint default occurs whenever all banks in the system default at the end of period  $T$ . More formally, given the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , the systemic default probability of the banking system,  $\mathbb{P}S$ , consisting of  $M$  banks  $i \in [1, M]$  is defined as:

$$\mathbb{P}S(f_1, n_1^*; f_2, n_2^*; \dots; f_M, n_M^*) := \Phi_M(z_1, z_2, \dots, z_M, \rho_{1,2}, \rho_{1,3}, \dots) \tag{11}$$

$\Phi_M$  is the standard  $M$ -variate cumulative normal distribution function with all pairwise asset correlations  $\rho_{i,j}$  included, where  $\rho_{i,j}$  is the asset correlation between banks  $i$  and  $j$ .

In the general case with  $M$  heterogeneous banks, Eq. 11 is hard to analyse when  $\rho_{i,j} = \rho_{i,j}(n_i^*, n_j^*)$  and  $z_i = z_i(n_i^*)$ . Numerical techniques also increase considerably in difficulty as the dimensionality of the problem grows. In choosing how to approach Eq. 11, we were driven by one main consideration – “what is the minimum amount of model complexity needed to study the interplay between diversification, leverage, and systemic risk?”. This is an important question, because there is always a trade-off between complexity and tractability. I.e. the more complex a model is, the harder it becomes to disentangle (either analytically or numerically) the effects of each individual model parameter on the result. Therefore, we make three simplifying assumptions. First, we confine our analysis to two banks, i.e.  $M=2$ . Second, we assume banks have the same position in the leverage-diversification space, i.e.  $(f_1, n_1^*) = (f_2, n_2^*) = (f, n^*)$ . Third, we assume the expected growth to be zero,  $\mu = 0$ . Systemic effects of a “booming” market,  $\mu > 0$ , compared to those of a “crashing” market,  $\mu < 0$ , are discussed in Section 4. Further in Section 4, we show that restricting ourselves to two homogeneous banks is sufficient model complexity to address our research question, with the added benefit of simplified analytical and numerical analysis.

With this in mind, the systemic default probability for two homogeneous banks becomes:

$$\mathbb{P}S(f, n^*) := \Phi_2(z_1, z_2, \rho_{1,2}), \quad \text{where } z_1 = z_2 \tag{12}$$

The calculation of Eq. (12) implies that we have to integrate the underlying density function  $g(z_1, z_2, \rho_{1,2})$  over a two-dimensional grid  $(z_1, z_2)$ , to obtain the cumulative function  $\Phi_2$ . This is generally not simple, especially when the asset correlation  $\rho_{1,2}$  is non-constant. Various tables exist if  $g$  is a bi-variate normal distribution (Pearson, 1931; Owen, 1962). Because this does not apply to our case, we use a method proposed by Jantaravareerat (1998), which allows us to calculate  $\Phi_2$  as follows:

1. Calculate the joint probability density function  $g(z_1, z_2, \rho_{1,2})$  defined as:

$$g(z_1, z_2, \rho_{1,2}) = \frac{1}{2\pi\sqrt{1-\rho_{1,2}^2}} \exp\left[-\frac{z_1^2 - 2\rho_{1,2}z_1z_2 + z_2^2}{2(1-\rho_{1,2}^2)}\right] \tag{13}$$

where the  $z_i$ 's are given in Eq. (10). This returns the joint probability density for a square grid with a side length,  $N$ , spanned by the range of  $z_1$  and  $z_2$ .

2. Map the sorted values of  $z_1, z_2$  to the index sets  $I_{z_1}$  and  $I_{z_2}$ , i.e. the elements of  $I_{z_1}$  and  $I_{z_2}$  are  $y_1$  and  $y_2$  that range from  $[1, 1]$  to  $[N, N]$ . Hence, after calculating the density  $g(z_1, z_2, \rho_{1,2})$ , we map it to  $g[y_1, y_2]$ <sup>15</sup> by using the row and column coordinates  $[y_1, y_2]$ .
3. Calculate the mean density  $d[y_1, y_2]$  at each cell as the mean of itself and its three immediate neighbours.

$$d[y_1, y_2] = \frac{1}{4}(g[y_1, y_2] + g[y_1 + 1, y_2] + g[y_1, y_2 + 1] + g[y_1 + 1, y_2 + 1]) \tag{14}$$

4. Calculate the volume of each cell,  $P[y_1, y_2]$  by multiplying the average density by the area, to accommodate for the fact that the indexed cells were originally of unequal size:

$$P[y_1, y_2] = d[y_1, y_2](z_2 - z_1)^2 \tag{15}$$

5. Calculate the joint distribution function,  $\Phi_2[y_1, y_2]$  for each cell  $[y_1, y_2]$  by summing up the volumes of all cells up to this point:

$$\Phi_2[y_1, y_2] = \sum_{j_1=1}^{y_1} \sum_{j_2=1}^{y_2} P[j_1, j_2] \tag{16}$$

6. Map  $\Phi_2[y_1, y_2]$  back to  $\Phi_2(z_1, z_2, \rho_{1,2})$  by using the index sets  $I_{z_1}, I_{z_2}$

In the following section we analyse Eq. (16), to study the systemic default probability w.r.t. the banks' position in the leverage-diversification space,  $(f, n^*)$ . Furthermore, in the homogeneous case the “overlapping correlation” in Eq. (8) becomes a constant expressed as:

$$\rho_{1,2} = \frac{n^{*2} \sigma^2}{N n^{*2}} \bigg/ \frac{\sigma^2}{n^*} = \frac{n^*}{N} := \rho \tag{17}$$

This reduces the problem to the discussion of  $\Phi_2(z, \rho)$  with  $z(f, n^*, \chi)$  given by Eq. (10).

#### 4. Analysis

In this section we adopt a *policy maker's* perspective to quantify the impact of banks' position in the leverage-diversification space  $(f, n^*)$  on the systemic default probability  $\Phi_2(z(f, n^*, \chi), \rho)$ , given some properties of the market,  $\chi$ . In detail, we want to assess the benefits of diversification to eliminate the portion of systemic risk that is caused by increased leverage. The latter is defined as the difference between two different leverage levels  $\Delta f := f_h - f_l$ , with  $f_h > f_l$ . We then calculate the difference  $\Delta\Phi_2(f_h, f_l) := \Phi_2(f_h) - \Phi_2(f_l)$ , which gives us the increase in systemic risk associated with  $\Delta f$ , all other parameters fixed. The goal is to find the critical level of diversification  $\hat{n} = \hat{n}(f_l, f_h, N, \chi)$  such that:<sup>16</sup>

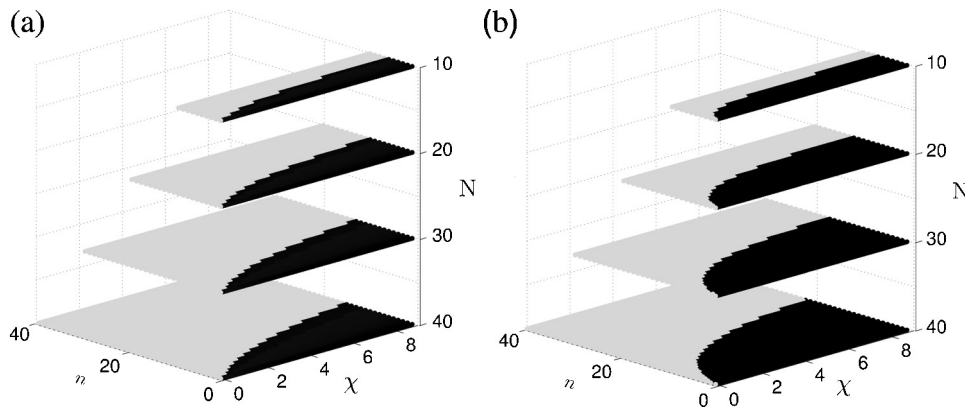
$$\begin{cases} \forall n \in [0, \hat{n}), \Delta\Phi_2(n) > 0 \\ \forall n \in [\hat{n}, N], \Delta\Phi_2(n) = 0 \end{cases}$$

$\hat{n}$  is the minimum level of diversification that banks should be forced to adopt in order to offset  $\Delta\Phi_2$ . We use the term “forced”

<sup>14</sup> Indirect effects such as price movements caused by the default of other banks are not considered here.

<sup>15</sup> Square brackets shall denote that we use the index instead of the actual values as coordinates

<sup>16</sup>  $\Delta\Phi_2(n) < 0$  is not possible, since increasing leverage should not result in decreasing risk.



**Fig. 1.** Effects of market size, degree of diversification, and leverage on the systemic default probability. The two plots show the difference  $\Delta\Phi_2(f_h, f_l)$ . **Left:**  $\{f_l, f_h\} = \{0.10, 0.25\}$  and  $\Delta f = 0.15$ . **Right:**  $\{f_l, f_h\} = \{0.25, 0.50\}$  and  $\Delta f = 0.25$ . Each plane corresponds to a particular market size,  $N$ . Black color indicates positive values of  $\Delta\Phi_2$ , whereas gray color indicates zero. Other parameters:  $N \in \{10, 20, 30, 40\}$ ,  $n \in [1, N]$ ,  $\chi \in [0.001, 9]$ .

**Table 1**

Minimum level of diversification  $\hat{n}(\chi)$  to eliminate the additional systemic risk due to increased leverage.

*Left:*  $\{f_l, f_h\} = \{0.10, 0.25\}$  and  $\Delta f = 0.15$ . *Right:*  $\{f_l, f_h\} = \{0.25, 0.50\}$  and  $\Delta f = 0.25$ . Other parameters:  $N \in \{10, 20, 30, 40\}$ ,  $\chi \in \{1.6, 5.1, 8.9\}$ . The numbers for  $\hat{n}$  are rounded to the nearest integers.

$N$	$\{f_l, f_h\} = \{0.10, 0.25\}$			$\{f_l, f_h\} = \{0.25, 0.50\}$		
	$\hat{n}(1.6)$	$\hat{n}(5.1)$	$\hat{n}(8.9)$	$\hat{n}(1.6)$	$\hat{n}(5.1)$	$\hat{n}(8.9)$
10	3	5	6	5	6	7
20	4	8	10	8	11	12
30	5	10	13	10	15	17
40	5	11	15	12	18	22

because this level may not coincide with  $n^*$ , especially when  $n^*$  is very small due to high transaction costs.

In Fig. 1 we plot  $\Delta\Phi_2$  over a parameter sweep of the market size,  $N$ , the market risk,  $\chi$ , and the number  $n \in [0, N]$  of possible projects to hold in the portfolio. Fig. 1 demonstrates the impact of the parameters  $N$ ,  $n$ , and  $\chi$  on the systemic risk for two different sets of leverage values:  $\{f_l, f_h\} = \{0.10, 0.25\}$  with  $\Delta f = 0.15$  and  $\{f_l, f_h\} = \{0.25, 0.50\}$  with  $\Delta f = 0.25$ . In the graph, we distinguish between  $\Delta\Phi_2 = 0$  (gray area<sup>17</sup>) and  $\Delta\Phi_2 > 0$  (black area). From this numerical analysis we can draw the following conclusions.

**Diversification:** For all other values constant, there exists a diversification level  $\hat{n}$  beyond which  $\Delta\Phi_2 = 0$ . Remarkably,  $\hat{n} \ll N$  (see the lines that separate the black from the gray areas in Fig. 1). In other terms, conditional on the market size  $N$  and market properties  $\chi$ ,  $\hat{n}$  represents the *minimum* number of risky projects each bank should possess in order to prevent an increase in systemic risk. As Fig. 1 indicates,  $\hat{n}$  increases monotonically with market size  $N$ , i.e., a bigger market requires more diversification. Moreover, we can show that  $\hat{n}$  increases for “crashing” markets,  $\mu < 0$ , and decreases for “booming” markets,  $\mu > 0$ . This is in line with economic intuition, i.e. diversification works well only in a booming market, (Tasca and Battiston, 2014). The specific dependence  $\hat{n}(f_l, f_h, N, \chi)$  can be obtained only numerically. Table 1 gives some numbers for the minimum value, in accordance with Fig. 1.

**Market size:** We can observe, both from Fig. 1 and Table 1, that for an increasing market size  $N$ , (i) the *absolute* number of projects necessary to offset  $\Delta\Phi_2$  increases, and (ii) the *relative* number of projects decreases. The first result is seemingly counter intuitive,

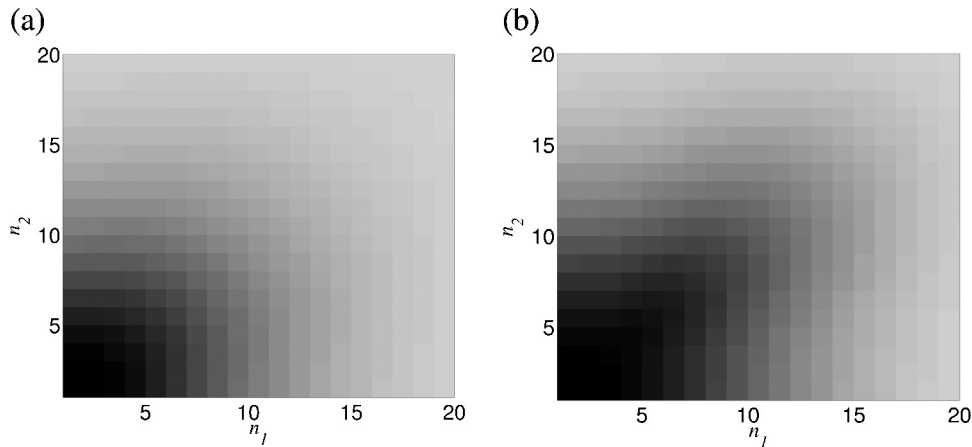
<sup>17</sup> We talk about small positive values in the order of  $10^{-6}$  or smaller, which are approximately zero

as one would expect that broader markets in which banks can choose from a bigger pool of projects allow for better diversification possibilities. However, larger markets reduce the effectiveness of diversification because they bear more risks which have to be compensated. In this context, systemic risk can be hardly contained through diversification, i.e., it is possible that  $n^* < \hat{n}$ . In Appendix A, we show analytically that the number of projects required to eliminate the same level of risk increases with the market sizes  $N$ . It could be more interesting, however, to discuss the second result related to the *relative risk*, i.e. the ratio of the black area (where systemic risk has increased with leverage increase) over the gray area (where systemic risk is independent of leverage and diversification). It appears that the relative risk *decreases* with increasing market size. i.e., for larger markets it becomes more likely to invest in projects such that banks 1 and 2 become independent enough with respect to their diversification strategy. For  $n^*$  randomly chosen projects, a larger market reduces the risk of systemic default, provided that  $n^*$  is above the critical number  $\hat{n}$ .

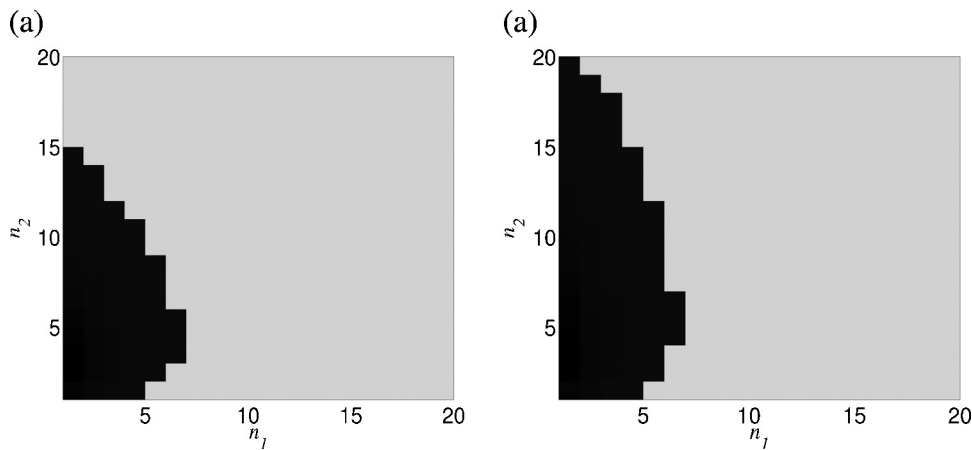
**Leverage and market risk:** Comparing both sides of Fig. 1, we observe that the more we increase leverage ( $\Delta f = 0.15$  vs  $\Delta f = 0.25$ ), the larger the black area, i.e.,  $\hat{n}$  increases monotonically with leverage increments. This also holds true for increasing absolute leverage values (not shown). However, it is worth noting that the impact of leverage is not as severe as one might expect. Its influence becomes noticeable mainly for higher values of  $\chi$ , i.e. for extreme market risks. Instead, in less volatile markets, i.e. for moderate market risks, the relative impact of leverage decreases.

**Correlation:** *Ceteris paribus*, the correlation between bank assets has a negative impact on systemic risk. This can be shown if we isolate the effect of the correlation on  $\Phi_2$  by considering two fixed (i.e., independent from the level of diversification) values of correlation, namely  $\rho_l$  and  $\rho_h$ , with  $\rho_l < \rho_h$ . Fig. 2 shows that for low levels of diversification, the systemic risk corresponding to  $\rho_h$  is higher than the systemic risk corresponding to  $\rho_l$ , i.e.,  $\Delta\Phi_2(\rho_h, \rho_l) := \Phi_2(\rho_h) - \Phi_2(\rho_l) > 0$  (black area in Fig. 2). However, by increasing diversification, the difference  $\Delta\Phi_2(\rho_h, \rho_l)$  decreases up to the point where it vanishes, i.e.,  $\Delta\Phi_2(\rho_h, \rho_l) = 0$  (gray area in Fig. 2). Since in our model  $\rho$  is an increasing function of  $n^*$ , full diversification (i.e.,  $n^* \rightarrow N$ ) exerts two effects on systemic risk. A first *direct* effect is positive because  $\Phi_2$  decreases with  $n^*$ . A second *indirect* effect is negative because  $\rho$  increases with  $n^*$ , and, in turn,  $\Phi_2$  increases with  $\rho$ . However, the direct positive effect of  $n^*$  exerted on  $\Phi_2$  is *stronger* than its indirect effect. Eventually, the negative impact of  $\rho$  on  $\Phi_2$  vanishes with diversification.

The main message drawn from the above consideration is the emergence of two market regimes: the *safe* regime in which the



**Fig. 2.** Isolated effect of correlation on the systemic default probability. The two plots show how the difference  $\Delta\Phi_2(\rho_h, \rho_l)$  converges to zero by increasing the level of diversification. **Left:**  $\rho_l = 0.10, \rho_h = 0.50$ . The difference  $\Delta\Phi_2(\rho_h = 0.50, \rho_l = 0.10)$  is positive (black area) for low levels of diversification. It vanishes for increasing levels of diversification (grey area). **Right:**  $\rho_l = 0.10, \rho_h = 0.90$ . The difference  $\Delta\Phi_2(\rho_h = 0.90, \rho_l = 0.10)$  is positive (black area) for low/medium levels of diversification. It vanishes for increasing levels of diversification (grey area). Other parameters:  $N = 20, n \in [1, N], \chi = 5$ .



**Fig. 3.** Effects of heterogeneous leverage between bank 1 and bank 2 on the systemic default probability. The two plots show the difference  $\Delta\Phi_2(f_1, f_2)$  for  $f_2 > f_1$ . **Left:**  $\{f_1, f_2\} = \{0.10, 0.25\}$  and  $\Delta f = 0.15$ . **Right:**  $\{f_1, f_2\} = \{0.10, 0.75\}$ . Black color areas indicate  $\Delta\Phi_2 > 0$ , whereas gray color areas indicate  $\Delta\Phi_2 = 0$ . Other parameters:  $N = 20, n \in [1, N], \chi = 5$ .

increased leverage does affect systemic risk (gray areas) and the risky regime in which systemic risk is indeed higher (black areas). If  $n^* < \hat{n}$ , there are two complementary ways to move from the risky to the safe regime. First, for a given market condition  $\chi$ , banks should increase, e.g. forced by regulations, their level of diversification at least up to  $\hat{n}$  (outward movement along the  $n$  axis in Fig. 1). Second, for a given diversification level  $n^* (< \hat{n})$ , the market should become less risky (inward movement along the  $\chi$  axis in Fig. 1).

As we mentioned in the previous section, our two-bank homogeneous model is a special case of a banking system with  $M$  banks heterogeneously positioned in the leverage-diversification space. An extension of the model in either direction is certainly possible, albeit at decreased analytical and numeric tractability. For example, Cathcart and El-Jahel (2004) analyze a static setting of  $M$  homogeneous banks where diversification, leverage and pairwise asset correlations are fixed arbitrarily and not dependent on each other.<sup>18</sup>

Here, we give an example of additional insights that may be gained if banks were heterogeneously positioned in the leverage-diversification space. In this case, the critical level of diversification would no longer be a scalar but a point in a  $M$ -dimensional vector space with the following coordinate:  $\hat{n} = (\hat{n}_1, \hat{n}_2, \dots, \hat{n}_M)$ . Intuitively, if some banks increase their level of leverage compared to others, systemic risk should increase, similarly to what we have observed earlier in the homogeneous case. From a policy maker's perspective this, in turn, generates negative externalities for the whole system because all banks (not only those highly leveraged) become more exposed to the risk of a systemic collapse. To better grasp this idea, let us consider two banks (bank 1 and bank 2) both having the same level of leverage, i.e.,  $f_1 = f_2 = f_l$ , but different diversification levels, i.e.,  $n_1^* \ll n_2^*$ . To these levels corresponds a certain systemic risk  $\Phi_2(f_1 = f_l, n_1^*; f_2 = f_l, n_2^*)$ , all other parameters fixed.

Now, if bank 2 increases its leverage to a higher level,  $f_2 = f_h > f_l$ , the systemic risk increases. Namely,  $\Delta\Phi_2(f_1 = f_l, f_2 = f_h) := \Phi_2(f_1 = f_l, n_1^*; f_2 = f_h, n_2^*) - \Phi_2(f_1 = f_l, n_1^*; f_2 = f_l, n_2^*)$  is the increase of systemic risk caused by bank 2' higher leverage (see the black areas in Fig. 3). In this case, the critical level of diversification is given by the

<sup>18</sup> This simplification makes it possible to apply standard numerical algorithms.

bi-dimensional vector space  $\hat{\mathbf{n}} = (\hat{n}_1, \hat{n}_2)$ , which elements are the pairs  $(\hat{n}_1, \hat{n}_2)$  of points separating the black from the gray area. As shown in Fig. 3 a), if  $\Delta f := f_h - f_l$  is moderate, it would be sufficient to force bank 2 to adopt a sufficiently high level of diversification ( $\hat{n}_2 = 15$ ) in order to move the whole system into the safe regime (gray area), regardless of the diversification level chosen by the other bank 1. However, if  $\Delta f$  further increases, as shown in Fig. 3 b), then the critical level of diversification changes and also bank 1 should now be forced to reach a certain minimum level of diversification in order to move the system into the safe regime. If however,  $n_1^* < \hat{n}_1 < n_2^*$ , a tension arises between what is best for an individual bank and what is safest for the system as a whole. This circumstance generates a *regulator's dilemma*: should banks be allowed to minimise their individual risk or should the risk of the system as a whole be regulated?

## 5. Conclusions

Financial institutions such as banks use different strategies to run their businesses. In this paper, we consider two of them, namely leverage and diversification. The first one allows banks to increase their expected return on equity based on debt financed investments. The second one is used to mitigate losses from such investments. Hence, both strategies have a different impact on the default probability of a bank: leverage amplifies the default risk, while diversification reduces it. In a finite market banks cannot choose completely independent diversification strategies. Instead, their portfolios overlap with diversification, which correlates the default probability of individual banks.

In this paper, we are interested in the impact of these strategies on the systemic default probability, i.e. the probability of joint defaults. In particular, we answer the question if and how an increase of individual risk, due to higher leverage, leads to an increase of the systemic default probability, and if and how this additional systemic risk can be mitigated through individually optimal diversification. As the main result, we determine the boundary between two market regimes: a safe regime in which the systemic impact of higher leverage can be mitigated by better diversification, and a risky regime in which this is not possible resulting in additional systemic risk.

Our findings contribute to the ongoing discussions on the micro-macro tension in the regulatory framework for financial markets. The question, whether individually optimal diversification can offset the systemic risk begot by higher leverage, can be answered by means of our distinction between risky and safe conditions. This also extends to the question whether capital requirements should be lowered for institutions with more diversified investments. Here, we show that a critical level of diversification exists, which depends on the market size and the market conditions, and thus needs to be monitored to prevent a transitions toward the risky regime. However, this level does not generally coincide with the one that is individually optimal. Such insights can be used to improve macro-prudential regulations, such as the current Basel III risk-based framework, which is portfolio invariant, i.e., does not consider the role for diversification. As a conclusion from our results, portfolio invariance leads to a significantly higher maximum leverage ratio allowed for banks in the current regulatory framework, i.e. it can result in a much riskier financial system.

## Appendix A.

### Proof. Optimisation with transaction costs

The optimisation problem in (4) is equivalent to:

$$\begin{aligned} \min_{\omega_1, \dots, \omega_l} \quad & \frac{1}{2} \sigma_i^2 \\ \text{s.t.} \quad & \begin{cases} \mu_i \geq \pi \\ w_{il} \geq 0 \\ \sum_{l=1}^{n_i} \omega_{il} = 1 \end{cases} \end{aligned} \quad (\text{A.1})$$

where  $\pi$  is the expected rate of return bank  $i$  wishes the portfolio to select realises. In vector notation the problem is:

$$\begin{aligned} \min_{\omega_1, \dots, \omega_l} \quad & \frac{1}{2} \omega' \Sigma \omega \\ \text{s.t.} \quad & \begin{cases} \omega' \mu \geq \pi \\ \omega' \mathbf{e} \geq 0 \\ \omega' \mathbf{e} = 1. \end{cases} \end{aligned} \quad (\text{A.2})$$

where  $\mu := \{\mu_1, \dots, \mu_l\}$  is the vector of expected returns from the projects in portfolio,  $\Sigma$  is the variance-covariance matrix of the projects in portfolios, and  $\mathbf{e}$  denotes the vector of all ones. For convenience, we omit the sub-index  $i$  in the vector notations.

In the following, we extend the original mean-variance optimisation problem to account for the presence of *convex* transaction costs. Our adaptation of the model by Mitchell and Braun (2002) captures the feature that transaction costs are paid when a project is bought or sold and the transaction cost reduces the amount of that particular projects that is available in the bank portfolio. We introduce the following notations.  $\bar{\omega}$  is the vector of weights of projects in the existing portfolio of the bank before rebalancing;  $\mathbf{c}$  is the vector of transaction costs that are assumed to be proportional to the value of the projects bought or sold, with the convention that: (1)  $\mathbf{c}(\omega)$  is a convex function of  $\omega$ ; (2)  $\mathbf{c}(\mathbf{0}) = \mathbf{0}$ ; (3)  $\mathbf{c}(\omega) \geq \mathbf{0}$  for all  $\omega$ .  $\mathbf{u}$  denotes the vector of weights of projects bought to rebalance old portfolio;  $\mathbf{v}$  is the vector of weights of projects sold to rebalance old portfolio. Then,

$$\omega = \bar{\omega} + \mathbf{u} - \mathbf{v} \quad (\text{A.3})$$

is the vector of weights of projects in the new portfolio after rebalancing. We let  $\omega_0$  denote the total amount spent on transaction costs:

$$\omega_0 = \mathbf{c}'(\mathbf{u} + \mathbf{v})$$

Exploiting the fact that  $\omega' \mathbf{e} = 1$ , Eq. (A.3) immediately gives:

$$(\mathbf{c} + \mathbf{e})' \mathbf{u} + (\mathbf{c} + \mathbf{e})' \mathbf{v} = 0$$

This equation can be used in place of the constraint used in (A.2) to give a model for minimising the variance of the resulting portfolio subject to meeting an expected return of  $\pi > 0$  in the presence of proportional transaction costs. The resulting model is:

$$\begin{aligned} \min_{\omega_1, \dots, \omega_l} \quad & \frac{1}{2} \omega' \Sigma \omega \\ \text{s.t.} \quad & \begin{cases} \omega' \mu \geq \pi \\ \omega = \bar{\omega} + \mathbf{u} - \mathbf{v} \\ (\mathbf{c} + \mathbf{e})' \mathbf{u} + (\mathbf{c} + \mathbf{e})' \mathbf{v} = 0 \\ \omega, \mathbf{u}, \mathbf{v} \geq 0 \end{cases} \end{aligned} \quad (\text{A.4})$$

When there are no transaction costs to be paid, one dollar is always available for investment, i.e.  $\sum_l \omega_l = 1$ . This assumption is implicit



in the standard mean-variance risk measure. However, for nonzero transaction costs that implicit assumption does not hold true. That is, one dollar is not longer available for investment, costs will be paid to rebalance. The appropriate value function  $V$  is therefore:

$$V := \frac{\frac{1}{2}\omega' \Sigma \omega}{(1 - \omega_0)^2} \tag{A.5}$$

where  $(1 - \omega_0)$  is the actual amount available for investment after having paid the transaction costs  $\omega_0$ . So we scale the standard risk measurement by the square of the dollar amount actually invested. This originate the “fractional quadratic programming problem” (FQP) (see Mitchell and Braun (2002)) as follows:

$$\begin{aligned} \min_{\omega_1, \dots, \omega_l} & \frac{\frac{1}{2}\omega' \Sigma \omega}{(1 - c'(u + v))^2} \\ \text{s.t.} & \begin{cases} \omega' \mu \geq \pi \\ \omega = \bar{\omega} + u - v \\ (c + e)u + (c + e)v = 0 \\ \omega, u, v \geq 0 \end{cases} \end{aligned} \tag{A.6}$$

Mitchell and Braun (2002) shows that if  $\omega^*$  is the solution of mean-variance optimisation problem (A.2), then

$$\hat{\omega}^* := \frac{\omega^*}{t^*}$$

is the solution of the FQP in (A.4) where  $t^* = 1/(1 - c'(u^* + v^*))$ . Since  $u$  and  $v$  are constrained to be non-negative,  $t \geq 0$  we have:

$$\hat{\omega}^* < \omega^*.$$

In Section 2.2 we assumed  $\mu_l = \mu$ ,  $\sigma_l = \sigma$  and  $\rho_{ly} = 0$ , such that the solution of (4), or equivalently (A.2), is  $\omega^* := \{(1/N), \dots, (1/N)\}$ . If now we apply the same assumptions to (A.4) the result of the optimisation problem will be  $\hat{\omega}^* := \{(1/(Nt^*)), \dots, (1/(Nt^*))\}$  that we can rewrite as:

$$\hat{\omega}^* := \{1/n^*, \dots, 1/n^*\}$$

where  $n^* := Nt^* < N$ . □

**Proof.** Critical diversification level  $\hat{n}$  increases with  $N$  and  $\chi$

Since, Eq. (12) shows that systemic risk is an increasing function of the idiosyncratic risk – amplified by  $\sigma^2$  in Eq. (10) – the seemingly counter intuitive result that  $\hat{n}$  increases with  $N$  can be explained by looking at the number  $n$  of projects required to eliminate a certain level of idiosyncratic risk for different market sizes  $N$ . (For convenience, in the following we omit the sub-index  $i$  in the notations). According to Eq. (6), the volatility of the bank assets is  $\sigma^2/n$ . For a portfolio with size  $n$ , the fraction  $1 - 1/n = (n - 1)/n$  of  $\sigma^2$  is the maximum idiosyncratic risk that can be eliminated with diversification. This level has to be compared with the total maximum idiosyncratic risk that can be eliminated with diversification if the portfolio size is  $N$ . Hence, we define the proportion  $\alpha$  of relative maximum idiosyncratic risk that can be eliminated with a portfolio of size  $n$  for a given market of size  $N$ :

$$\alpha = \frac{1 - 1/n}{1 - 1/N} = \frac{N n - 1}{n N - 1} \tag{A.7}$$

For a given market size  $N$ ,  $\bar{n}(\alpha)$  is then the number of projects to hold in order to eliminate  $\alpha\%$  of the idiosyncratic risk. Ideally,  $\alpha$  should be set above the 95% level. Table 2 shows that the minimum number  $\bar{n}(\alpha)$  of projects, required to eliminate the same level  $\alpha$  of idiosyncratic risk, increases with the market size.

□

**Table 2**

According to Eq. (A.7) the table presents the number of minimum required projects to hold in order to eliminate  $\alpha\%$  of the diversifiable risk. The figures are rounded to the nearest integers.

N Market size	Level $\alpha$ of idiosyncratic risk to eliminate		
	$\alpha = 0.95$	$\alpha = 0.98$	$\alpha = 0.99$
	Number $\bar{n}(\alpha)$ of required projects		
10	7	8	9
20	10	14	17
30	12	19	23
40	14	22	29

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