Advances in Complex Systems, Vol. 13, No. 4 (2010) 483–499
© World Scientific Publishing Company
DOI: 10.1142/S0219525910002700

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FROM ASSORTATIVE TO DISSORTATIVE NETWORKS: THE ROLE OF CAPACITY CONSTRAINTS

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Received 25 March 2010
Revised 1 July 2010

We consider a dynamic model of network formation where agents form and sever links based on the centrality of their potential partners. We show that the existence of capacity constrains in the amount of links an agent can maintain introduces a transition from dissortative to assortative networks. This effect can shed light on the distinction between technological and social networks as it gives a simple mechanism explaining how and why this transition occurs.

1. Introduction

Networks represent connections existing among individuals, firms and institutions. Connections may comprise friendship ties, financial exchanges, risk sharing, collaboration between economic agents in technological areas, exchange of information, trade agreements, conversation, familial relations, co-membership in associations, joint presence at events, etc. Network analysis examines the implications of these patterns for social, political and technological processes. In all these settings, the outcome of agents depends on the structure of the network.

Key characteriztics of real-world social and economic networks are:

(1) A small average shortest path length between any pair of agents [1].
(2) A high clustering, which means that the neighbors of an agent are likely to be connected [52].
(3) An inverse relationship between the clustering coefficient of an agent and her degree [20, 43]. The neighbors of a high degree agent are less likely to be connected among each other than the neighbors of an agent with low degree. This

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means that empirical networks are characterized by a negative clustering-degree correlation.

(4) A highly skewed degree distribution. While some authors [7] find power law degree distributions, others find either deviations from power laws, like e.g., [42], or exponential distributions [23].

(5) Degree-degree correlations for economic networks. [40,41] has shown that many *social networks* tend to be positively correlated. In that case, the network is said to be *assortative*. On the other hand, *technological networks* such as the internet [43] display negative correlations. In that case, the network is said to be *dissortative*. Others, however, find negative correlations in social networks such as the Ham radio network consisting of interactions between amateur radio operators [32] or the affiliation network in a Karate club [53]. Networks in economic contexts may have features of both technological and social relationships [30]. Indeed, there exist networks with positive degree correlations such as the venture capitalist one [38] as well as negative degree correlations as in the world trade web [46], online social communities [28] and bank networks [16,39].

To fathom these different aspects and to match the observed structure of real-life networks, one has to analyze how and why networks form and what are the mechanisms that describe their evolution over time.

We are interested in two different approaches describing the emergence of networks. On the one hand, there are models describing network formation in a purely stochastic way (mostly developed by mathematicians and physicists): networks are either growing through the sampling of a stochastic process and links appear at random according to some distribution, or built according to some algorithm. In the other approach (mostly developed by economists), the reason for the formation of a link lays on strategic interactions. Individuals carefully decide with whom to interact and this decision entails some consent by both parts in a given relationship (see [29,30] for a complete overview of these two approaches). There is also another literature, which we will refer to as "games on networks," which takes the network as given and studies how the network structure impacts on outcomes and individual decisions. A prominent paper of this literature is [6]. Their main finding is that if agents' pay-offs are linear-quadratic and embedded in a network, then the unique interior Nash equilibrium of the corresponding n-player game, is such that each individual effort is proportional to the Bonacich centrality measure, a well-known measure in sociology introduced by [10].[a]

[a]In fact, *centrality* is a fundamental measure of the importance of actors in social networks and its importance was already stressed in early works such as [8]. See [51] for a complete introduction and survey of this literature. In many situations, it is the centrality of an agent in a network that explains her outcome and decisions. In the empirical literature, it has been shown that centrality is important in explaining exchange networks [15], peer effects [12,17,26], creativity of workers [44], flow of information [11], formation and performance of R&D collaborating firms and inter-organizational networks [9,48] as well as the success of open-source projects [22].

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In a previous paper [34], we have combined all these three different approaches (e.g. random and strategic network formation as well as games on networks) to characterize the stationary distribution of emerging networks. In this model, agents form and severe links by basing their decisions on the centrality of their potential partners. We have considered different types of centrality measures and have shown that the dynamics of network formation as well as the stationary distribution do not depend on the type of centrality measure considered. One of the main results of [34] is to show that the networks that emerge at any moment of time (and of course at the steady state) are nested-split graphs. These networks have properties which can also be found in many real-world networks. In particular, the stationary networks (which are nested split graphs) are characterized by *short path length*, *high clustering*, a *power-law tail* in the degree distribution and *dissortativity*.

The aim of the present paper is to extend [34] by introducing a new mechanism that may explain the emergence of *assortativity* in social networks. We demonstrate that the existence of *capacity constrains* in the amount of links an agent can maintain leads to assortative networks. We show that if agents still decide their link formation and deletion in a strategic way, e.g. based on the centrality of the possible partners, but are constrained in the number of links they can maintain, then emerging stationary networks are characterized by positive degree-degree correlations and thus assortativity. This effect may shed some light on the distinction between technological and social networks as suggested by [40, 41]. Following our findings, technological networks are facing capacity constraints to a much lower extent than social networks. Indeed, consider the internet, a prominent example of a technological network and the e-mail network in an organisation, a prototype of a social network. The number of hyper-links a website can contain may not be limited as much as the number of social contacts (measured e.g. by mutual email exchange) an individual in an organisation may keep. Thus, the distinction between technological and social networks and the degree of assortativity and degree-degree correlations can be derived from the severity of capacity constraints imposed on the number of links an agent can maintain.

We note however, that there may exist exceptions to the above distinction between social and technological networks in terms of capacity constraints. The neighborhood size might be larger in a social network than in a technological network, depending on the application and how one classifies a network according to these categories. However, we restrict our argument to cases in which nodes in a social networks have smaller neighborhood size (e.g. in mutual email communication networks [23], networks of social acquaintances [53] or coauthorship networks [20]) than nodes in a technological network (e.g. the internet [43]).

We also note that there have been a number of alternative explanations for the presence of degree-degree correlations. [31] show that dissortativity (prevalent in technological networks) is the result of a maximum entropy principle. Similarly, [14] propose a network growth model which is able to generate assortative networks. Their model is a variant of the preferential attachment model [7], where a network

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grows over time by successively adding links originating from new nodes added to the network, with the characteriztic that links are also formed between already existing nodes. The authors show that this mechanism is able to generate assortative networks. Differently to these authors, we show that assortativity can be the result of capacity constraints in the number of links an agent can maintain.

This paper is organized as follows. In Sec. 2, we review the original model of [34] and its main results. Section 3 introduces the model of capacity constrains and the results are discussed in Sec. 4. Finally, Sec. 5 concludes.

2. The Model of [34]

In [34], we consider a network $G = (V, E)$ composed of a set V of n agents and a set E of m links. We assume that initially, at time $t = 0$, the network is empty. Then, at time $t = 1$, an agent is chosen at random and with probability $\alpha \in [0, 1]$ she can form a link. Because she is indifferent, this agent creates a link with any other agent in the network. At time $t = 2$, again, an agent is chosen at random and with probability α, decides with whom she wants to form a link while with probability $1 - \alpha$, this agent has to delete a link if she has already one. And so forth. In this framework, the randomly chosen agent does not create or delete a link randomly. On the contrary, she calculates all the possible network configurations and chooses to form (delete) a link with the agent that gives her the highest utility (reduces the least of her utility). It turns out that connecting to the agent with the highest Bonacich centrality (deleting the link with the agent that has the lowest Bonacich centrality) is a best-response function for this agent. In [34], there is a game in efforts that rationalizes this behavior. This is exactly the game developed by [6].

To summarize, the dynamics of network formation is as follows: at time t, an agent i is chosen at random. With probability α, agent i creates a link to the most central agent while with complementary probability $1-\alpha$ agent, i removes a link to the least central agent in her neighborhood. Then, time is increased by $t \to t + 1/n$.

Importantly, the dynamics of this network formation is not tied to a particular prescription of the centrality. Indeed, the network formation process remains the same if instead of forming links to agents with highest Bonacich centrality, they create links with agents that have the highest degree, closeness, eigenvector or betweenness[b] centralities.

In [34], it is shown that, at every period, the emerging network is a *nested split graph* [3] or a *threshold network* [25, 36], whose matrix representation is stepwise. This means that agents can be rearranged by their degree *rank* and agents with degree d are connected to all agents with degrees larger than d. Moreover, if two agents i, j have degrees such that $d_i < d_j$, this implies that their neighborhoods

[b]Assuming that agents with the same betweenness centrality are ranked according to another centrality measure, for example their degree.

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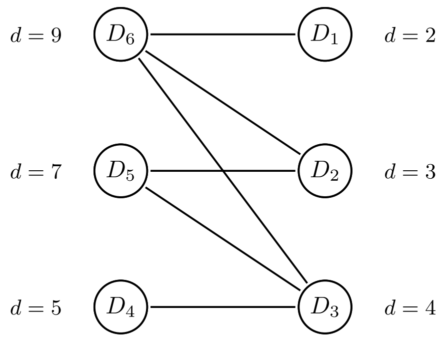

$$\mathbf{A} = \begin{pmatrix} 0&1&1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1&0&0\\ 1&1&1&0&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Fig. 1. Representation of a connected nested split graph G (left) and the associated adjacency matrix $\mathbf{A}$ (right) with $n = 10$ agents and $K = 6$ distinct degrees. D_i, $1 \leq i \leq 6$ denotes the set of nodes in G with the ith smallest degree. A line between D_i and D_j indicates that every node in D_i is linked to every node in D_j. Next to the set D_i the degree of the nodes in the set is indicated. In the corresponding adjacency matrix $\mathbf{A}$ to the right the zero-entries can be separated from the one-entries by a step function.

satisfy $\mathcal{N}_i \subset \mathcal{N}_j$. An illustration is given in Fig. 1. The nested neighborhood structure indicates that our networks are strongly hierarchical. An important property of nested split graph is that the diameter is two, e.g. the maximum distance between two agents in a nested split graph is two. This is because all agents are connected to the one that has the largest number of links. For our network formation process, this implies that, in a nested split graph, agents can only form links to second-order neighbors, e.g. neighbors of neighbors.

The stepwise property of the adjacency matrix $\mathbf{A}$, with elements $a_{ij} \in \{0, 1\}$, during the network evolution can be easily found by induction: At time $t = 0$, the first link addition generates a (trivial) stepwise matrix. Next, assume that this is true at time t. We now consider the creation of a link ij. In the case of using the eigenvector centrality,[c] let λ_{PF} be the largest (Perron–Frobenius) eigenvalue [27,45] and $\mathbf{v} = \{v_i\}_{1 \leq i \leq n}$, the associated non-negative eigenvector of $\mathbf{A}$. Then

$$v_i = \frac{1}{\lambda_{\mathrm{PF}}} \sum_{j=1}^{n} a_{ij} v_j = \frac{1}{\lambda_{\mathrm{PF}}} \sum_{j \in N_i} v_j. \tag{1}$$

It follows that the larger the neighborhood N_i (degree) of agent i, the higher its eigenvector component v_i. This means that the eigenvector centrality of the agents is ranked in the same way as their degree (centrality). The same property can be trivially shown for closeness centrality: agents with the largest degree in a stepwise matrix are closer to the rest; and betweenness centrality: agents with the largest degree are part of a larger amount of geodesic paths connecting two other agents. Therefore, a link is created to the agent with the highest degree not already connected to agent i. This preserves the stepwise property of $\mathbf{A}$. Similarly, agent i will

[c]Since the dynamic network formation process is invariant to any centrality measure, this is only to facilitate the presentation. In [34], the proof is performed for the Bonacich centrality game.

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severe the link to the agent with the lowest degree in agent i's neighborhood and, therefore, the stepwise property of $\mathbf{A}$ is maintained.

Given the symmetry in the adjacency matrix $\mathbf{A}$, in order to solve the dynamic evolution of the network, it is enough to solve the dynamics for the agents with a degree smaller or equal than $K/2$, when there are K distinct degrees in the network. Denote by $N(d,t)$ the number of agents with degree $d \leq K/2$ at time t. Starting from an empty network, it can be shown that the dynamic evolution is given by[d]:

$$N(d,t'+1) - N(d,t') = \frac{1-\alpha}{n}N(d+1,t') + \frac{\alpha}{n}N(d-1,t') - \frac{1}{n}N(d,t'), \quad (2)$$

$$N(0,t'+1) - N(0,t) = \frac{1-2\alpha}{n} - \frac{\alpha}{n}N(0,t) + \frac{1-\alpha}{n}N(1,t). \quad (3)$$

These equations mean that the probability to add nodes to the class with degree d is proportional to the number of nodes with degree $d-1$ (resp. $d+1$) when selected for node addition (deletion). The dynamics of the adjacency matrix (and from this the complete structure of the network) can be directly recovered from the solution of these equations.

For this purpose, we analyze here the continuous limit for large networks. We perform the following change of variables: $t'/n \to t$; $d/n \to k$, and then, k is defined in the interval $[0,1]$; $\delta t = 1/n$, $\delta k = 1/n$, and $z(k,t) = N(d,t')/n$. In this limit, Eqs. (2) and (3) read [33]:

$$\partial_t z(k,t) = (1-2\alpha)\,\partial_k z(k,t) + \delta k\,\partial^2_{kk} z(k,t) + \mathcal{O}(\delta k^2), \quad (4)$$

$$\begin{aligned}\partial_t z(0,t) &= (1-2\alpha)(\delta k + \partial_k z(0,t) - \alpha\,\delta k\,z(0,t)) + \mathcal{O}(\delta k^2),\\ z(1,t) &= 0 + \mathcal{O}(\delta k^2).\end{aligned} \quad (5)$$

The initial condition corresponds to an empty graph with $z(k,0) = \delta(0)$. We have included the terms of order δk and higher orders (which vanish for infinite networks), because they will play a central role in the dynamics of the system.

First, when such terms can be neglected, Eq. (4) becomes a usual deterministic drift equation whose stationary solution is a complete network, $z(d) = 1$ if $\alpha > 1/2$; or $z(k) = 0$ if $\alpha < 1/2$, e.g. an empty network. This result explains that, when the link decay is low ($\alpha > 1/2$), the agents can keep the connections, and the overall density of the network is high. On the other hand, as we will show below, when the decay is high, the network rapidly converges to a hierarchical structure where only a few agents rapidly become central, and without major exogenous disturbances, they remain in this central position forever. Initial stochastic influences and path dependency are the deciding factors that determine who will be central. Similar to [2] the competition driven network dynamics leads to the spontaneous emergence of hubs.

[d]We have assumed that for all agents with degree $d < K/2$ the difference in the degree of an agent and the agent with the next higher degree is one.

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The first order transition in the network density gives rise to non-trivial effects around the critical point $\alpha = 1/2$. At this point define, $1 - 2\alpha = \beta\,\delta k$. If $\beta \sim \mathcal{O}(1)$, the diffusion term in Eq. (4) is not negligible any more and the boundary conditions become a simple reflecting boundary. Time scales must be reduced once more, so by now rescaling it into $\tau \equiv t\,\delta k$, we get the following Fokker–Planck equation [19]:

$$\begin{aligned} \partial_\tau z(k,\tau) =& \beta\partial_k z(k,\tau) + \partial_k^2 z(k,\tau), \\ \partial_\tau z(0,\tau) =& \beta\partial_k z(0,\tau), \end{aligned} \tag{6}$$

with the same reflecting boundary conditions as in the complete problem. This prescription allows to relate the width of the transition from sparse to dense networks: $\beta \sim \mathcal{O}(1)$, e.g. it must be of the order of one; this implies that $\Delta\alpha/\delta k = (1-2\alpha)/\delta k \sim \mathcal{O}(1)$ and it follows that the width of the transition scales as $\Delta\alpha \sim n^{-1}$.

Let us now determine the stationary solutions for all values of $\alpha \in [0,1]$. First, notice that the network G obtained for a value of $\alpha > 1/2$ is the complement of the network obtained for $1-\alpha$. Thus, in the following we consider only values of $\alpha \leq 1/2$. Let $h(x)$ be the continuous limit of the step function in the adjacency matrix $\mathbf{A}$ (see Fig. 1), rescaled such that $x = 1-k \in [0,1]$. $h(x)$ can be decomposed in a part $h_u(x)$ below the diagonal and a part $h_l(x)$ above the diagonal of $\mathbf{A}$. The point x^* is implicitly defined by $h_u(x^*) = h_l(x^*)$, where the step-function intersects with the diagonal. Let $z(k) = \lim_{\tau\to\infty} z(k,\tau)$ be the stationary degree distribution. We have that $h_u(x) = \int_0^{1-x} z(k)dk$. From the stationary solution of (6), we find

$$h_u(x) = \mathcal{N}e^{-2(1-2\alpha)x}, \tag{7}$$

with the constant

$$\mathcal{N} = \frac{2(1-2\alpha)}{1-e^{-2(1-2\alpha)n}}, \tag{8}$$

and $\lim_{\alpha\to 1/2}\mathcal{N} = 1/n$.

This result for the functional form of the step-function is valid for the elements below the diagonal, e.g. for the agents with low degree. We now turn our attention to the high degree, central agents. From the symmetry of the adjacency matrix, it is easily seen that $h_l(x)$ for these agents satisfies the following equation

$$x = \mathcal{N}e^{-2(1-2\alpha)h_l(x)}. \tag{9}$$

Thus, inverting this expression we get

$$h_l(x) = \frac{\ln(\mathcal{N}) - \ln(x)}{2(1-2\alpha)} \tag{10}$$

Conversely, the degree distribution is given by $z(k) = -h'(1-k)$ and we obtain in the stationary state

$$z(k) = \begin{cases} \mathcal{N}e^{-2(1-2\alpha)k}, & \text{if } k < 1-x^*, \\ \dfrac{1}{2n(1-2\alpha)}k^{-1}, & \text{if } k > 1-x^*. \end{cases} \tag{11}$$

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For $\alpha = 1/2$, we obtain a uniform distribution $z(k) = 1/n$. Note that we generate power-law tails in the degree distribution with an exponent -1. Thus we are able to produce power law degree distributions in a model without network growth, differently to e.g. the preferential attachment model by [7].

Since we are able to compute the adjacency matrix, all network statistics of interest can be readily computed. In particular, one can show that the stationary networks emerging from our link formation process are characterized by *short path length* (at most two) with *high clustering* (so-called "small worlds"), *exponential degree distributions* with *power law tails* and *negative degree-clustering correlations*. Moreover, we find that stationary networks are *dissortative*. Also, there exists a phase transition at $\alpha = 1/2$ from highly centralized to highly decentralized networks. This means that for low arrival rates of linking opportunities α (and a strong link decay), the stationary network is strongly polarised, composed mainly of a star, while for high arrival rates of linking opportunities (and a weak link decay), stationary networks are dense and largely homogeneous. We also find that the transition between these states is sharp and becomes sharper with increasing system size.

3. Capacity Constraints and Global Search

A natural generalization of the model discussed so far is to allow for the possibility that agents do not accept to establish a link with another agent that wants to connect to them. The underlying assumption is that agents face *capacity constraints* in the number of links they can maintain. Such constraints can arise from a possible information overload and congestion [4, 18, 24]. Compared to the model of [34], if they are capacity constrained, agents can now refuse a link-creation proposal whereas before it was always beneficial to accept it. If a link-creation proposal has been refused, then the selected agent will not only search among her neighbors' neighbors but also among all agents in the network (random search). However, agents preferably connect to their neighbors' neighbors and, if this fails, they search for new contacts at random. This means that, if capacity constraints prevent an agent from forming a link locally, we assume that she tries to link to an agent out of the whole population at random. This mechanism introduces a *global search mechanism* in the link formation process (see [37, 49] for a similar approach). One of the main consequences of introducing capacity constraints and random search in [34] is that networks are not nested split graphs anymore. This is mainly due to the random search aspect of the process, which eliminates the very hierarchical structure of nested split graphs since now agents with low degrees can be asked to form a link.

By introducing capacity constraints and random search in [34], we find that stationary networks become now *assortative*, a feature that was not possible in the original model. As a result, the emergence of assortativity and positive degree-correlations, respectively, can be explained by considering limitations in the

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number of links an agent can maintain. This may be of particular relevance for social networks and give an explanation for the distinction between assortative social networks and dissortative technological networks, as suggested by [40].

We assume that capacity constraints arise from the fact that an agent can only interact with one other agent at a time. Each neighbor requests information with probability β. Assuming that information requests are independent, the probability that an agent $i \in N$ with d_i links does not receive any information requests from her neighbors is given by $(1-\beta)^{d_i}$. If an agent does not receive such an information request, then she can accept an additional link; otherwise she will not.

Moreover, we allow for the formation of links between agents that are not connected through a common neighbor. This means that agents search globally for new contacts (see also [50]) if they cannot connect to the agent with the highest centrality among their neighbors' neighbors. When an agent i is selected, she tries to connect to the agent j with the highest degree in her neighborhood. However, agent $j \in \mathcal{N}_i^{(1)}$ only accepts the link formation with probability $(1-\beta)^{d_j}$. Otherwise, agent i selects another agent $k \in N \backslash \{\mathcal{N}_i^{(1)} \cup i \cup j\}$ out of the whole population of agents (excluding agents i and j) uniformly at random, and this link also has the same acceptance probability $(1-\beta)^{d_k}$ based on the degree of agent k.[e]

With probability α, a randomly selected agent i creates a link with the agent j who has the highest centrality among her second order neighbors. This occurs with probability $(1-\beta)^{d_j}$. With probability $(1-(1-\beta)^{d_j})\sum_{k \in N \backslash \{\mathcal{N}_i^{(1)} \cup i \cup j\}}(1-\beta)^{d_k}$, agent i forms a link at random to another agent out of the whole population. As before, with probability $1-\alpha$, agent i deletes a link to the agent with the lowest centrality.

We make the following technical assumption. In the model exposed in Sec. 2, the eigenvector centrality of an agent increases the most if she forms a link to the agent with the highest degree. For the current model, we assume that this property is still approximately true. In most cases, this approximation can be made but there exist exceptions in which the degree and Bonacich centrality ranking do not coincide [21].

More formally, we define the network formation process $(G(t))_{t=0}^{\infty}$, $G(t) = (N, L(t))$, as a sequence of networks $G(0), G(1), G(2), \ldots$ in which at every step $t = 0, 1, 2, \ldots$, an agent $i \in N$ is uniformly selected at random. Then one of the following two events can occur:

(1) With probability $\alpha \in (0,1)$, agent i receives the opportunity to create an additional link. Let j be the agent in $\mathcal{N}_i^{(2)}$ with the highest degree, that is $d_j \geq d_k$ for all $j, k \in \mathcal{N}_i^{(2)}$. Then with probability $(1-\beta)^{d_j}$, the link ij is formed.

[e] Let $\mathcal{N}_i = \{k \in N : ik \in L(t)\}$ be the set of neighbors of agent $i \in N$ and $\mathcal{N}_i^{(2)} = \bigcup_{j \in \mathcal{N}_i} \mathcal{N}_j \backslash (\mathcal{N}_i \cup \{i\})$ denote the second-order neighbors of agent i in the current network $G(t)$. Note that the connectivity relation is symmetric such that j is a second-order neighbor of i if i is a second-order neighbor of j, i.e. $i \in \mathcal{N}_j^{(2)}$ if and only if $j \in \mathcal{N}_i^{(2)}$ for all $i, j \in N$.

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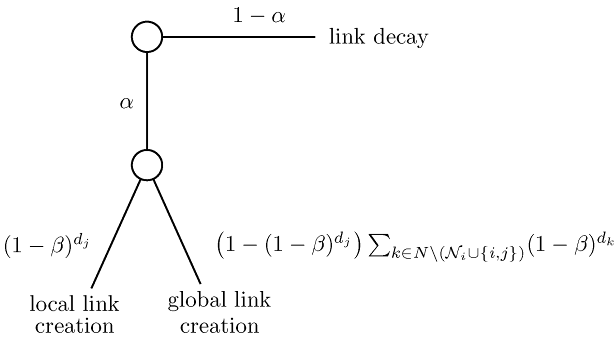

Fig. 2. Probabilities with which a randomly selected agent i creates a link and a link of agent i decays, respectively, when capacity constraints are taken into account (assuming that the agent is neither isolated nor fully connected).

Otherwise agent i connects to a randomly selected agent, $k \in N\backslash(\mathcal{N}_i \cup \{i, j\})$, with probability $(1-(1-\beta)^{d_j})(1-\beta)^{d_k}$. If agent i is already connected to all other agents, then nothing happens.

(2) With probability $1-\alpha$, the link to the agent j in $\mathcal{N}_i$ with the smallest degree $d_j \le d_k$ for all $j, k \in \mathcal{N}_i$, decays. If agent i does not have any links, then nothing happens.

An illustration of the above link formation process $(G(t))_{t=0}^{\infty}$ is shown in Fig. 2. An agent, i, is selected at random and either creates a link or the link to the neighbor with lowest degree decays with probability $1-\alpha$. However, with probability α, agent i is selected to create a link. In this case, agent i forms the link to agent j with the highest degree among her second-order neighbors with probability $(1-\beta)^{d_j}$ and to another agent out of the whole population of agents at random with probability $(1-(1-\beta)^{d_j})\sum_{k\in N\backslash(\mathcal{N}_i\cup\{i,j\})}(1-\beta)^{d_k}$.

4. Results

Having introduced the extended network formation process, we now investigate its properties by means of computer simulations for values of $\alpha \in [0.2, 0.5]$ and $\beta \in [0.01, 1]$. We consider a set of $n = 1000$ agents and use a sample of 30 to 40 simulation runs from which we compute the average as an approximation to the stationary network.

Figure 3 shows the clustering and assortativity of stationary networks for different values of α and β. We find that for values of β around 0.1 and in $\alpha \in [0.45, 0.5]$, stationary networks are *assortative* while displaying a high clustering (albeit lower than in the basic model without capacity constraints). It is relatively easy to understand why networks become assortative. Indeed, if α is high but not too high (links are formed at a relative high rate), while β is quite low (meaning that $1-\beta$ is high so that agents do accept link proposals), then low-degree agents will be connected to low-degree agents, a feature not possible in the model without capacity constraints

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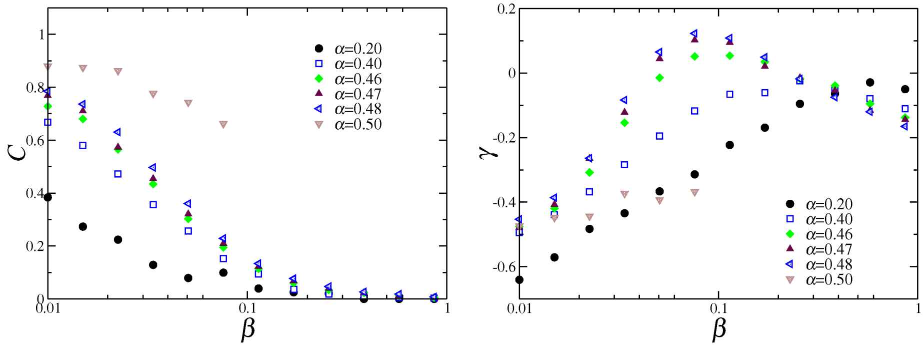

Fig. 3. In the left panel, we show the clustering coefficient obtained by recourse of numerical results of the extended model with capacity constraints for different values of α and β in a system with $n = 1000$ agents. In the right panel we show the corresponding network assortativity. Each different curve corresponds to a different value of α. Only agents that are not isolated are considered.

(in a nested split graph, low-degree agents cannot be connected to low-degree agents).

The characteristic path length $\mathcal{L}$ is defined as the number of links in the shortest path between two agents, averaged over all pairs of agents [52]. This can be written as

$$\mathcal{L} = \frac{1}{n(n-1)} \sum_{u \neq v} d(u,v), \tag{12}$$

where $d(u,v)$ is the geodesic (shortest path) between agent u and agent v. Taking the inverse of the shortest path length, one can introduce a related measurement, the network efficiency, $\mathcal{E}$, that is also applicable to disconnected networks [35]

$$\mathcal{E} = \frac{1}{n(n-1)} \sum_{u \neq v} \frac{1}{d(u,v)}. \tag{13}$$

In Fig. 4, we show the characteriztic path length $\mathcal{L}$ and the efficiency $\mathcal{E}$ measuring shortest paths in the network. The plots indicate that stationary networks in the extended model exhibit short path lengths between the agents.[f] We find that the stationary network may not just consist of one connected component and possibly isolated nodes, but it may have multiple components. However, there exists a giant component encompassing at least 90% of the nodes in all the simulations we studied.

We further analyze the degree distribution of stationary networks and we find that it is highly skewed following an exponential function.

[f] The average path lengths generated by our model are at most six, as it is shown in Fig. 4. Real-world networks have average path lengths that are typically larger than two. However, there exist real world networks with very short average path lengths. Examples are the network of entrepreneurs in Silicon Valley studied by [5, 13] or the network of banks analyzed by [47], which has an average path length of around 2.6.

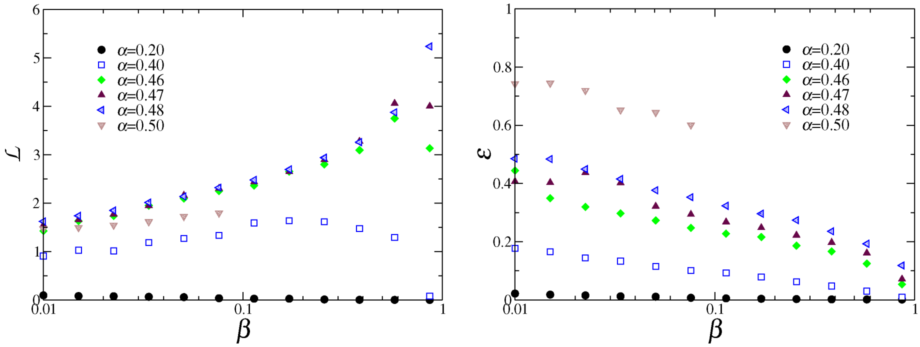

Fig. 4. We show the measures for the network topology obtained by recourse of numerical results of the model with capacity constraints for different values of α and β in a system comprised of $n = 1000$ agents. The left panel shows the characteriztic path length $\mathcal{L}$ of the network G^* and the right panel shows the results for the network efficiency $\mathcal{E}$.

Moreover, we find that the results for different centralization measures show a similar behavior as we have seen already in the original model. There exists a sharp, albeit less pronounced, transition from highly centralized networks to homogeneous networks by increasing α above 1/2.

In Fig. 5, we show the fraction of the largest real eigenvalue of the stationary network. The largest real eigenvalue is a measure of efficiency of the network, e.g. the network that maximizes total welfare (see [34]). For values of $\alpha < 1/2$, stationary networks are highly inefficient with respect to the complete network while the extent of inefficiency can be drastically reduced if one takes the reduced network density into account.

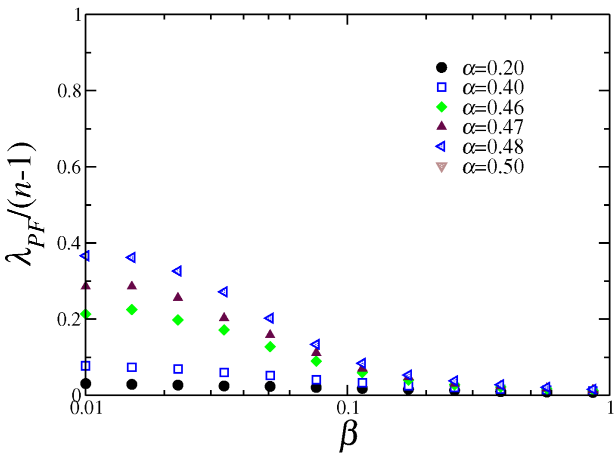

Fig. 5. We show the largest eigenvalue of the adjacency matrix normalized to the largest one in a complete graph (which is the efficient network [6]), obtained by recourse of numerical results for the model with capacity constraints for different values of α and β in a system comprised of $n = 1000$ agents.

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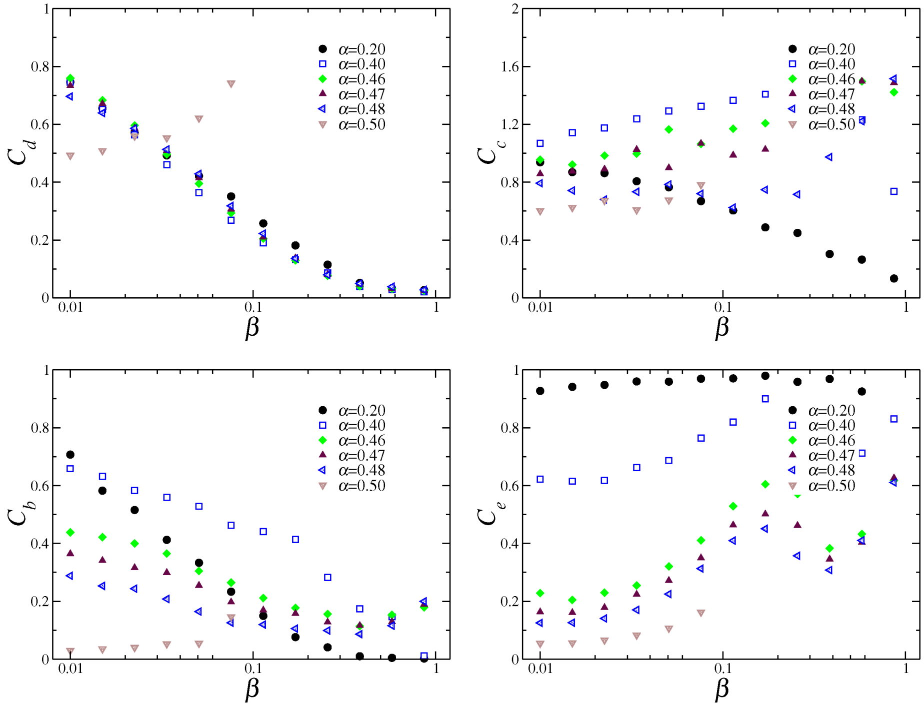

Fig. 6. Degree, closeness, betweenness and eigenvector centralization in the stationary networks for different values of α, in the constrained model. For all centralization measures we obtain a sharp transition between strongly centralized networks for lower values of α and decentralized networks for higher values of α.

Finally, Fig. 6 shows centralization in the stationary network for different values of β. This figure reveals that stationary networks tend to be highly centralized for low values of β. This indicates that, in this parameter range, stationary networks are highly unequal and characterized by few central agents.

In this section, we have studied different network statistics for different values of α and β. We find that, by introducing capacity constraints and global search, stationary networks become *assortative* while exhibiting an *exponential degree distribution*, *high clustering*, *short average path length* and *negative clustering-degree correlation*. These characteriztics can be found in social and economic networks as well. Thus, our model is able to reproduce the main characteriztics of real world networks to the whole extent, ranging from assortative to dissortative networks.

5. Concluding Remarks

In this paper, we have introduced a network formation process with capacity constrains in which link creation and removal are based on the position of the agents in the network as measured by their centrality.

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In the original model of [34], when agents have only local information when forming links and their connections are exposed to a volatile environment, the emergence of hierarchies depends on the level of volatility of the environment. Two different regimes appear: (i) when linking opportunities are rare, the network rapidly converges to a hierarchical structure where only a few agents rapidly become central; (ii) when linking opportunities abound, flat structures arise. It is also found that there exists a sharp transition in the core-periphery structure of the network from a highly centralized to a decentralized network.

In the present paper, we consider the role of *capacity constraints* and *global search* in this network formation process. We find that stationary networks become *assortative* while exhibiting an exponential degree distribution, high clustering, short average path length and negative clustering-degree correlation. These characteriztics can be found in social and economic networks as well. Thus, our model is able to reproduce characteriztics of real world networks, ranging from assortative to dissortative networks.

Our findings have an implication for the distinction between assortative and dissortative networks. As discussed in the preceding sections, our network formation process generates stationary networks that are characterized by negative degree-degree correlation and dissortativity. On the other hand, capacity constraints transform stationary networks to exhibiting positive degree-degree correlations and assortativity. This effect may shed some light on the origin of the distinction between technological and social networks suggested by [40,41], where technological networks are characterized by dissortativity and social networks by assortativity. Following our findings, technological networks are facing capacity constraints to a much lower extent than social networks. Consider, for example, the internet as a technological network and the email network in an organization as a prototype of a social network. The number of hyper-links a website can contain may not be limited as much as the number of social contacts (measured e.g. by mutual e-mail exchange) an individual in an organization may keep. Thus, the distinction between technological and social networks and the degree of assortativity and degree–degree correlations can be derived from the severity of capacity constraints imposed on the number of links an agent can maintain.

Acknowledgments

We are grateful to the editor and two anonymous referees for helpful comments.

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