Chapter 8
Modeling Evolving Innovation Networks

Michael D. König, Stefano Battiston and Frank Schweitzer

8.1 Introduction

8.1.1 The Importance of Innovation Networks

Economists widely agree on technological change and innovation being the main components of economic growth (Aghion and Howitt, 1998; Tirole, 1988). In the absence of ongoing technological improvements, economic growth can hardly be maintained (Barro and Sala-i Martin, 2004). The close link between innovation and economic performance has become generally accepted. Following this insight, in recent years of economic growth, OECD countries have fostered investments in science, technology, and innovation (OECD, 2006).

Moreover, technologies are becoming increasingly complex. This increasing complexity of technologies can make an agent's "in-house" innovative effort insufficient to compete in an R&D intensive economy. Thus, agents have to become more specialized on specific domains of a technology and they tend to rely on knowledge transfers from other agents, which are specialized in different domains, in order to combine complementary domains of knowledge for production ("recombinant growth" (Weitzman, 1998)).

When one agent benefits from knowledge created elsewhere we speak of knowledge spillovers. Knowledge spillovers define "any original, valuable knowledge generated somewhere that becomes accessible to external agents . . . other than the originator"[1] (Foray, 2004, p. 95).

The knowledge-based economy is developing towards a state in which the costs for acquiring, reproducing, and transmitting knowledge are constantly decreasing,

M.D. König (✉)
Chair of Systems Design, ETH Zurich, Kreuzplatz 5, CH-8092 Zurich, Switzerland
e-mail: mkoenig@ethz.ch

[1] "Involuntary spillovers are a feature of market competition. Competition not only creates incentives to produce new knowledge but it also forces the other agents to increase their own performance through imitation, adoption and absorption of the new knowledge created elsewhere" (Foray, 2004, p. 91).

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spatial and geographical limitations on knowledge exchange are becoming less important, and attitudes change towards more open behavior of sharing knowledge instead of hiding it from others. In this state, knowledge externalities will play an increasingly important role.

When agents are using knowledge that is created elsewhere, they must have access to other agents across a network whose links represent the exchange or transfer of knowledge between agents. The importance of networks in innovative economies has been widely recognized, e.g., it has been observed that "the development of knowledge within industries is strongly influenced by the network structure of relations among agents" (Antonelli, 1996, p. 1). Subsequently, an ample body of empirical research has documented the steady growth of R&D partnerships among firms (Hagedoorn et al., 2006).

8.1.2 Markets for Knowledge Exchange

The exchange of knowledge is not unproblematic. Markets for knowledge exchange can exhibit serious market failures (Arora et al., 2004; Gerosky, 1995), which make it difficult for innovators to realize a reasonable return from trading the results of their R&D activities (the problem of appropriability (Gerosky, 1995)). This is due to the public good character of knowledge, which makes it different from products or services. Knowledge is non-rival, meaning its use by one agent does not diminish its usability by another agent, and sometimes (when knowledge spillovers cannot be avoided) non-excludable, meaning that the creator of new knowledge cannot prevent non-payers from using it. The problems associated with trading of knowledge can prevent agents from exchanging knowledge at all.

There are three generic reasons for failures of markets for technology (Arora et al., 2004; Arrow, 1962; Gerosky, 1995): (1) economies of scale/scope, (2) uncertainty and (3) externalities.

(1) R&D projects often require huge initial investments and they can exhibit economies of scale since the cost for useful technological information per unit of output declines as the level of output increases (Wilson, 1975). Besides, Nelson (1959) has shown that economies of scope can apply to innovative agents. The broader an agents' "technological base," the more likely it is that any outcome of its R&D activities will be useful for her. The result is that markets for knowledge exchange are often dominated by monopolies.
(2) Almost all economic investments bear a risk of how the market will respond to the new product (commercial success). Innovators face additional risks. First, their investment into R&D does not necessarily lead to a new technology. Second, if such a new technology is discovered, it has to be put into practice in a new and better product than the already existing ones. This inherent uncertainty of R&D projects often causes agents to invest "too little."
(3) Externalities are important when the action of one agent influences the profits of another agent without compensation through the market. Public goods are a

typical example of creating externalities. Knowledge is a public good and the returns innovators can realize often are far below their investments into R&D. This can seriously diminish an agent's incentive to do R&D.

In order to overcome the above mentioned problems associated with the returns on investment into R&D, appropriate incentive mechanisms have to be created that encourage agents to invest into R&D. In general, Von Hippel and Von Krogh (2003) suggest three basic models of encouraging agents to invest into R&D:

(1) The private investment model assumes that innovation is undertaken by private agents investing their resources to create an innovation. Society then provides agents with limited rights to exclusively use the results of their innovation through patents or other intellectual property rights (by creating a temporarily monopoly).[2]
(2) The so-called collective action model (Allen, 1983) assumes that agents are creating knowledge as a public good. Knowledge is made public and unconditionally supplied to a public pool accessible to everybody. The problem is that potential beneficiaries could wait until others provide the public good and thereby could free-ride. One solution to this problem is to provide contributors (in this case innovators) with some form of subsidy. Scientific research is such an example where reputation-based rewards are granted to scientists for their good performance.
(3) In the private-collective innovation model participants use their private resources in creating new knowledge and then make it publicly available. This is typically observed in open-source projects. There are several incentives (Lerner and Tirole, 2002; Von Hippel and Von Krogh, 2003) for agents to participants in open-source projects. These range from elevated reputations, the desire of building a community to the expectation of reciprocity from the community members for their efforts.

The collective action approach (2) gives a possible explanation for the willingness of agents to share knowledge if there are no costs associated with it. One can think of a pool of technologies that is accessible to everybody ("broadcasting" of technologies) (Allen, 1983). This can be the case where agents are non-rivals and shared information may have no competitive cost. Additionally, knowledge must be easily understandable and transferable. This assumes that knowledge is highly codified[3] such that the transfer of knowledge from one agent to the other is costless. But, if these assumptions do not hold, the costs for transferring knowledge can

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[2] For a more detailed treatment of this issue we recommend Scotchmer (2004).

[3] The opposite case of codified knowledge is tacit knowledge. "Tacit knowledge is difficult to make explicit for transfer and reproduction. The exchange, diffusion, and learning of tacit knowledge require those who have it to take deliberate action to share it. This is difficult and costly to implement . . . Knowledge can, however, be codified. It can be expressed in a particular language and recorded on a particular medium. As such, it is detached from the individual. When knowledge is codified, it becomes easily transferable" (Foray, 2004, p. 73).

often be considerable, and agents become more selective about whom to share their knowledge with. We study this situation in the next sections.

8.1.3 Economies as Evolving Networks

As we have already outlined above, modern economies are becoming increasingly networked, and this also affects the innovation process where information and knowledge are exchanged by interactions between agents (Gallegati and Kirman, 1999; Kirman, 1997). In the agent-based view, the aggregate behavior of the economy (macro-economics) cannot be investigated in terms of the behavior of isolated individuals. Not only there are different ways in which firms interact, learning over time, based on their previous experience; also interactions between them take place within a network and not in a all-to-all fashion.

The standard neoclassical model[4] of the economy assumes that anonymous and autonomous individuals take decisions independently and interact only through the price system, which they cannot influence at all. However, competition easily becomes imperfect because, if agents have only a minimal market power, they will anticipate the consequences of their actions and anticipate the actions of others.

Game theorists have tried to integrate the idea of strategically interacting agents into a neoclassical[5] framework. But still they leave two questions unanswered. First, it is assumed that the behavior is fully optimizing. This leads to agents with extremely sophisticated information processing capabilities. Such ability of passing these enormous amounts of information in short times cannot be found in any realistic setting of human interaction. Advances in weakening that assumption are referred to as "boundedly rationality" (Gigerenzer and Selten, 2002). Second, the problem of coordination of activities is not addressed in the standard equilibrium model of the economy. Instead it is assumed that every agent can interact and trade with every other agent, which becomes quite unrealistic for large systems.

One has to specify the framework within the individual agents take price decisions, and thus limit the environment within which they operate and reason. An obvious way is to view the economy as a network in which agents interact only with their neighbors. In the case of technological innovation, neighbors might be similar firms within the same industry, but these firms will then be linked either through customers or suppliers with firms in other industries. Through these connections innovations

[4] A standard neoclassical model includes the following assumptions (Gabszewicz, 2000): (1) perfect competition, (2) perfect information, (3) rational behavior, (4) all prices are flexible (all markets are in equilibrium). The resulting market equilibrium (allocation of goods) is then efficient. See Hausman (2003) for a discussion of these assumptions.

[5] The individual decision making process is represented as maximizing a utility function. A utility function is a way of assigning a number to every possible choice such that more-preferred choices have a higher number than less-preferred ones (Varian, 1996). The gradients of the utility function are imagined to be like forces driving people to trade, and from which economic equilibria emerge as a kind of force balance (Farmer et al., 2005).

will diffuse through the network. The rate and extent of this diffusion then depends on the structure and connectivity of the network. The evolution of the network itself should be made endogenous where the evolution of the link structure is dependent on the agents' experience from using the links available to them. In this framework the individuals learn and adapt their behavior and this in turns leads to an evolution of the network structure. The economy then becomes a *complex evolving network*.

8.1.4 Complex Networks

Although no precise mathematical definition exists for a *complex network*, it is worth to elaborate the notion associated with it. In general, a network is a set of items some of which are linked together by pairwise relationships. The structure of the relationships can be represented mathematically as a graph in which nodes are connected by links (possibly with varying strength). However, a network is usually also associated with some dynamic process on the nodes which in turn affects the structure of the relationships to other nodes. A wide variety of systems can be described as a network, ranging from cells (a set of chemicals connected by chemical reactions), to the Internet (a set of routers linked by physical information channels). It is clear that the structure of the relationships co-evolves with the function of the items involved.

As a first step, a network can be described simply in terms of its associated graph.[6] There are two extreme cases of relatively simple graphs: regular lattices on one side and random graphs[7] on the other side. During the last century, graph theory and statistical physics have developed a body of theories and tools to describe the behavior of systems represented by lattices and random graphs. However, it turns out that, at least for physical scales larger than biomolecules, most systems are not structured as lattices or as random graphs. Moreover, such a structure is not the result of a design, but it emerges from self-organization. In some cases self-organization results from the attempt to optimize a global function. In other cases, as it is typical in economics, it results from nodes locally trying to optimize their goals, e.g., an individual utility function.

Large networks are collectively designated as complex networks if their structure (1) is coupled to the functionality, (2) emerges from self-organization, and (3) deviates from trivial graphs. This definition includes many large systems of enormous technological, intellectual, social, and economical impact (Frenken, 2006).

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[6] In general, a graph represents pairwise relations between objects from a certain collection. A graph then consists of a collection of nodes and a collection of links that connect pairs of nodes.
[7] The classical Erdös-Reny random graph is defined by the following rules (Bollobas, 1985):

(1) The total number of nodes is fixed.
(2) Randomly chosen pairs of nodes are connected by links with probability p.

The construction procedure of such a graph may be thought of as the subsequent addition of new links between nodes chosen at random, while the total number of nodes is fixed.

8.1.5 The Statistical Physics Approach

As we will discuss later, many of the theoretical tools developed in economics and specifically in game theory to characterize the stability of small networks of firms cannot be used for large networks. Asking which is the optimal set of connections that a firm should establish with other firms has little meaning in a large network if strategic interaction is taken into account (with more than say, 100 nodes it is simply not feasible to compute). On the other hand, it makes sense to ask what are connectivity properties of the nodes a firm should try to target in order to improve its utility with a certain probability. It is then necessary to turn towards a statistical description of these systems, where one is no longer interested in individual quantities but only in averaged quantities.

There exists an arsenal of such tools developed within *statistical physics* in the last century that allow to predict the macroscopic behavior of a system from the local properties of its constituents (Durlauf, 1999; Amaral et al., 1999). Such tools work very well for systems of identical particles embedded in regular or random network structures in which interactions depend on physical distance. Both a regular and a random structure have a lot of symmetries, which one can exploit to simplify the description of the system. However, in complex networks many of those symmetries are broken: individuals and interactions are heterogeneous. Moreover the physical distance is often irrelevant (think for instance of knowledge exchange via the Internet). Therefore, a satisfactory description of such systems represents a major challenge for statistical physics (Amaral et al., 2001).

In the last few years, we have thus witnessed an increasing interest and effort within the field of statistical physics in studying complex networks that traditionally were object of investigation by other disciplines, ranging from biology to computer science, linguistics, politics, anthropology, and many others. One of the major contributions of statistical physics to the field of complex networks has been to demonstrate that several dynamic processes taking place on networks that deviate from random graphs, exhibit a behavior dramatically different from the ones observed on random graphs.

An example for all is the case of virus spreading: it has been shown that while for random networks a local infection spreads to the whole network only if the spreading rate is larger than a critical value, for scale-free networks[8] any spreading rate leads to the infection of the whole network. Now, technological as well as social networks are much better described as scale-free graphs than as random graphs. Therefore all vaccination strategies for both computer and human viruses, which have been so far designed based on the assumption that such networks were random graphs, need to be revised. This highly unexpected result goes against volumes previously written on this topic and is due to the presence of a few nodes with very

[8] A scale-free network is characterized by a degree distribution which follows a power-law, $f(d) = \alpha d^{\gamma}$. The degree distribution gives the number of agents with a certain number, d, of in- or outgoing links (in- or out-degree), see the next section for a definition of degree of a node.

large connectivity. In this case, the rare events (infection of highly connected nodes) and not the most frequent ones matter.

Explaining the macroscopic behavior of a system in terms of the properties of the constituents has been a major success of the physicist's reductionist approach. But, while in physical systems the forces acting on single constituents can be measured precisely, this is not the case in a socio-economic system where, moreover, each agent is endowed with high internal complexity. Today, the physicist's approach to socio-economic systems differs from the nineteenth century positivist approach in so far as it does not aim at predicting, for instance, the behavior of individual agents. Instead, taking into account the major driving forces in the interactions among agents at the local level we try to infer, at a system level, some general trends or behavior that can be confirmed looking at the data. This is also very different from taking aggregate quantities and infer a macroscopic behavior from a "representative" agent[9] as it is done in several approaches in mainstream economics.

8.1.6 Dynamics Versus Evolution in a Network

After discussing the notion of a complex network which has been strongly influenced by physics, we now try to classify different complex networks.

The nodes in an economic network are associated with a state variable, representing the agents' wealth, a firm's output or, in the case of innovation networks, knowledge. There is an important difference between the evolution of the network and the dynamics taking place on the state variables. In the first, nodes or links are added to/removed from the network by a specific mechanism and in the latter, the state variables are changed as a result of the interactions among connected nodes (see also Gross and Blasius, 2007 for a review). Consequently, there are four aspects that can be investigated in complex economic networks (Battiston, 2003).

1. Statistical characterization of the static network topology without dynamics of state variables,
2. network evolution without dynamics of the state variables,
3. dynamics of state variables in a static network and
4. dynamics of state variables and evolution of the network at the same time.

This can be incorporated in the following Table 8.1.

[9] The concept of the representative agent assumes an economy which consists of a sufficiently homogeneous population of agents. Because all the agents are equivalent, the aggregate quantities of the system can be calculated by multiplying the average agent, or the representative agent, by the number of agents (the system size). For example, the total production of an economy is obtained by summing up the production levels of the individual firms that constitute the economy. To determine the behavior of the system, it is therefore sufficient to know the characteristics of the representative agent.

Table 8.1 Overview of the different ways in which a network and the state variables of the nodes can be related

Case	State variables	Network
1.	Static	Static
2.	Dynamic	Static
3.	Static	Dynamic
4.	Dynamic	Dynamic

In socio-economic systems as well as in biological systems, dynamics and evolution are often coupled, but do not necessarily have the same time scale. In Sect. 8.3.9, we will show how the coupling of fast knowledge growth (dynamics) and slow network evolution can lead to the emergence of self-sustaining cycles in a network of knowledge sharing (cooperating) agents.

8.1.7 Outline of This Chapter

In this chapter, we focus on (i) the emergence and (ii) the performance of different structures in an evolving network. The different scenarios we develop shall be applied to firms exchanging knowledge in a competitive, R&D intensive economy. In the existing literature reviewed in the following section, there are two different lines of research addressing these problems: (i) Models of network formation were developed based on individual utility functions, e.g., by Jackson (2003), in which simple architectures emerge in the equilibrium. (ii) In another group of models, e.g., Padgett et al. (2003), firms have specific skills and take actions based on goals or learning, and innovation is associated with the emergence of self-sustaining cycles of knowledge production. Although both lines of work address the problem of network emergence and performance, they differ significantly in terms of methods and results. We try to bridge them by introducing a novel model of evolving innovation networks that combines the topological evolution of the network with dynamics associated with the network nodes.[10]

We start our approach by giving a short introduction to graph theory in Sect. 8.2.1. Here we restrict ourselves only to the most important terms and definitions that are necessary in the following sections.[11]

We then proceed by giving an overview of the existing literature on economic networks. In the first part of our literature review, we explore some basic models of innovation networks. The selection of these models is by no means unique nor exhaustive, but points to important contributions to the growing literature on economic and innovation networks.[12] Similar to our own approach, these models make considerably simple assumptions and thus allow for analytical insights. This holds

[10] The approach of combining a dynamics of the network with a dynamics in the nodes is discussed in Gross and Blasius (2007).

[11] The reader interested in a good introduction in graph theory can consult West (2001).

[12] For an excellent introduction, see Jackson (2008), Vega-Redondo (2007), and Goyal (2007).

in particular for the connections model in Sect. 8.2.2. The model in Sect. 8.2.3 can be considered as an extension of the basic connections model where "small-world" networks emerge. In the subsequent Sect. 8.2.4, we discuss a model that takes heterogeneous knowledge into account, as a further extension. In the second part of the literature review, we briefly sketch models in which we observe cyclic network topologies. We show that in certain cases the stability of a network and its performance depends critically on its cyclic structure. The critical role of cycles in a networked economy has already been identified by Rosenblatt (1957) and many succeeding authors, e.g., Bala and Goyal (2000), Kim and Wong (2007), Maxfield (1994). In this chapter, we review some recent models in which cyclic networks emerge: in Sect. 8.2.5, we first introduce a model of production networks with closed loops and next we discuss a model of cycles of differently skilled agents.

Finally, in Sect. 8.3 we develop a novel framework, which we call *Evolving Innovation Networks*, to study the evolution of innovation networks. We show how different modalities of interactions between firms and cost functions related to these interactions can give rise to completely different equilibrium networks. We have studied the case of linear cost and bilateral interactions in König et al. (2008a,b). There we find that, depending on the cost, the range of possible equilibrium networks contains complete, intermediate graphs with heterogeneous degree distributions as well as empty graphs. Here, we focus on a type of non-linear cost and both, on unilateral and bilateral interactions. In the unilateral case, we find that, in a broad range of parameter values, networks can break down completely or the equilibrium network is very sparse and consists of few pairwise interactions and many isolated agents. Equilibrium networks with a higher density can be reached if (i) the utility function of the agents accounts for a positive externality resulting from being part of a technological feedback loop or if (ii) all interactions are bidirectional (direct reciprocal). Otherwise, the network collapses and only few, if any, agents can beneficially exchange knowledge.

The results found in our novel approach to evolving innovation networks are summarized in Sect. 8.4.1. The appendices shall be useful for the reader interested in more numerical results and the parameters and explanation of the algorithms used.

8.2 Basic Models of Innovation Networks

8.2.1 Graph Theoretic Network Characterization

Before we start to describe specific models of economic networks, we give a brief introduction to the most important graph theoretic terms used throughout this chapter to characterize networks. For a broader introduction to graph theory see West (2001). In this chapter, we will use the terms graph and network interchangeably, i.e., both refer to the same object.

A *graph* G is a pair, $G = (V, E)$, consisting of a set of node $V(G)$ and a set of links $E(G)$. K_n is the *complete graph* on n nodes. C_n the *cycle* on n nodes. Nodes i and j are the endpoints of the link $e_{ij} \in E(G)$.

The *degree*, d_i, of a node i is the number of links incident to it. A graph can either be *undirected* or *directed*, where in the latter case one has to distinguish between *in-degree*, d_i^-, and *out-degree*, d_i^+, of node i. In the case of an undirected graph, the neighborhood of a node i in G is $N_i = \{w \in V(G) : e_{wi} \in E(G)\}$. The degree of a node i is then $d_i = |N_i|$. The first-order neighborhood is just the neighborhood, N_i, of node i. The second-order neighborhood is, $N_i \cup \{N_v : v \in N_i\}$. Similarly, higher order neighborhoods are defined. In the case of a directed graph, we denote the out-neighborhood of node i by N_i^+ and the in-neighborhood by N_i^-. A graph G is *regular* if all nodes have the same degree. A graph G is *k-regular* if every node has degree k.

A *walk* is an alternating list, $\{v_0, e_{01}, v_1, ..., v_{k-1}, e_{k-1k}, v_k\}$, of nodes and links. A *trail* is a walk with no repeated link. A *path* is a walk with no repeated node. The shortest path between two nodes is also known as the *geodesic distance*. If the endpoints of a trail are the same (a closed trail) then we refer to it as a *circuit*. A circuit with no repeated node is called a *cycle*.

A *subgraph*, G', of G is the graph of subsets of the nodes, $V(G') \subseteq V(G)$, and links, $E(G') \subseteq E(G)$. A graph G is *connected*, if there is a path connecting every pair of nodes. Otherwise, G is disconnected. The *components* of a graph G are the maximal connected subgraphs.

The *adjacency matrix*, $\mathbf{A}(G)$, of G, is the n-by-n matrix in which the entry a_{ij} is 1 if the link $e_{ij} \in E(G)$, otherwise a_{ij} is 0. For an undirected graph, $\mathbf{A}$ is symmetric, i.e., $a_{ij} = a_{ji}$ $\forall i, j \in V(G)$. An example of a graph and its associated adjacency matrix is given in Fig. 8.1. For example, in the first row with elements, $a_{11} = 0, a_{12} = 1, a_{13} = 0, a_{14} = 0$, the element $a_{12} = 1$ indicates that there exists a link from node 1 to node 2.

In a *bipartite* graph G, $V(G)$ is the union of two disjoint independent sets V_1 and V_2. In a bipartite graph, if $e_{12} \in E(G)$ then $v_1 \in V_1$ and $v_2 \in V_2$. In other words, the two endpoints of any link must be in different sets. The *complete bipartite graph* with partitions of size $|V_1| = n$ and $|V_2| = m$ is denoted $K_{n,m}$. A special case is the *star* which is a complete bipartite graph with one partition having size $n = 1$, $K_{1,m}$.

There exists an important class of graphs, *random graphs*, which are determined by their number of nodes, n, and the (independent) probability p of each link being present in the graph (Bollobas, 1985).

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

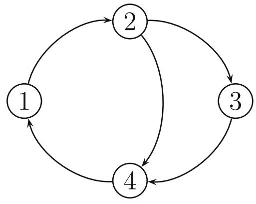

Fig. 8.1 (*Right*) A directed graph consisting of four nodes and five links. (*Left*) The corresponding adjacency matrix A

We now introduce two topological measures of a graph, the clustering coefficient and the average path length. For further details see, e.g., Newman (2003) and Costa et al. (2007). The following definitions assume undirected graphs.

For each node, the *local clustering coefficient*, C_i, is simply defined as the fraction of pairs of neighbors of i that are themselves neighbors. The number of possible neighbors of node i is simply $d_i(d_i - 1)/2$, where d_i is the degree of node i. Thus we get

$$C_i = \frac{|\{e_{jk} \in E(G) : e_{ij} \in E(G) \wedge e_{ik} \in E(G)\}|}{d_i(d_i - 1)/2}. \tag{8.1}$$

The *global clustering coefficient* C is then given by

$$C = \frac{1}{n} \sum_{i=1}^{n} C_i. \tag{8.2}$$

A high clustering coefficient C means (in the language of social networks), that the friend of your friend is also likely to be your friend. It also indicates a high redundancy of the network.

The *average path length*, l, is the mean geodesic (i.e., shortest) distance between node pairs in a graph:

$$l = \frac{1}{\frac{1}{2}n(n-1)} \sum_{i \geq j}^{n} d_{ij}, \tag{8.3}$$

where d_{ij} is the geodesic distance from node i to node j.

In Sect. 8.2.3, we will show a model of innovation networks that produces "small-worlds" which combine the two properties of a high clustering coefficient and a small average path length.

In the following sections, we will describe some basic models of economic network theory, where we shall use the definitions and notations introduced above.

8.2.2 The Connections Model

The *connections model* introduced by Jackson and Wolinsky (1996) is of specific interest, since it allows us to compute equilibrium networks analytically. The succeeding models can then be considered as extension of the connections model. Since these models are more complicated than the basic connections model they can, to a large extent, only be studied via computer simulations.[13] Nevertheless, they are of interest because they show a wider range of possible network configurations and associated performance of the agents in the economy.

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[13] For the use of computer simulations in economics, see Axelrod and Tesfatsion (2006).

In the following, we discuss the (symmetric) connections model proposed by Jackson and Wolinsky (1996).[14] In this model, agents pass information to other agents to whom they are connected to. Through these links they also receive information from those agents that they are indirectly connected to, that is, through the neighbors of their neighbors, their neighbors, and so on.

The individual incentives to form or severe links determine the addition or deletion of links. Incentives are defined in terms of the utility of the agents which depends on the interactions among agents, i.e., the network. The utility functions assigns a payoff to each agent and this payoff depends on the network the agents are embedded in.

The utility, $u_i(G)$, agent i receives from network G[15] with n agents is a function $u_i : \{G \in \mathcal{G}_n\} \to \mathbb{R}$ with

$$u_i(G) = \sum_{j \neq i} \delta^{d_{ij}} - \sum_{j \in N_i} c, \tag{8.4}$$

where d_{ij} is the number of links in the shortest path between agent i and agent j. $d_{ij} = \infty$ if there is no path between i and j. $0 \leq \delta < 1$ is a parameter that takes into account the decrease of the utility as the path between agent i and agent j increases. $N(i)$ is the set of nodes in the neighborhood of agent i. c is a positive parameter capturing the fact that direct links are costly. This implies that agents want to have short paths to other agents while maintaining as few links as possible.

A measure of the global performance of the network is introduced by its efficiency. The total utility of a network is defined by

$$U(G) = \sum_{i=1}^{n} u_i(G). \tag{8.5}$$

A network is considered efficient if it maximizes the total utility of the network $U(G)$ among all possible networks, $\mathcal{G}(n)$ with n nodes.

Definition 1 A network G is strongly efficient if $U(G) = \sum_{i=1}^{n} u_i(G) \geq U(G') = \sum_{i=1}^{n} u_i(G')$ for all $G' \in \mathcal{G}(n)$

Under certain conditions no new links are accepted or old ones deleted. This leads to the term pairwise stability.

Definition 2 A network G is pairwise stable if and only if

1. for all $e_{ij} \in E(G)$, $u_i(G) \geq u_i(E \backslash e_{ij})$ and $u_j(G) \geq u_j(E \backslash e_{ij})$
2. for all $e_{ij} \notin E(G)$, if $u_i(G) < u_i(E \cup e_{ij})$ then $u_j(G) > u_j(E \cup e_{ij})$

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[14] For a good introduction and discussion of related works, we recommend the lecture notes of Zenou (2006). There one can find the proofs given here and related material in more detail. For a general introduction to economic networks, see also Jackson (2006).

[15] In this model the network is undirected.

In words, a network is pairwise stable if and only if (i) removing any link does not increase the utility of any agent, and (ii) adding a link between any two agents, either does not increase the utility of any of the two agents, or if it does increase one of the two agents' utility then it decreases the other agent's utility.

The point here is that establishing a new link with an agent requires the consensus (i.e., a simultaneous increase of utility) of both of them. The notion of pairwise stability can be distinguished from the one of Nash equilibrium,[16] which is appropriate when each agent can establish or remove unilaterally a connection with another agent.

There exists a tension between stability and efficiency in the connections model. This will become clear, after we derive the following two propositions.

Proposition 3 *The unique strongly efficient network in the symmetric connections model is*

1. *the complete graph* K_n *if* $c < \delta - \delta^2$,
2. *a star encompassing everyone if* $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$,
3. *the empty graph (no links) if* $\delta + \frac{n-2}{2}\delta^2 < c$.

Proof 1. We assume that $\delta^2 < \delta - c$. Any pair of agents that is not directly connected can increase its utility (the net benefit for creating a link is at least $\delta - c - \delta^2 > 0$) and thus the total utility, by forming a link. Since every pair of agents has an incentive to form a link, we will end up in the complete graph, where all possible links have been created and no additional links can be created any more.

2. Consider a component of the graph G containing m agents, say G'. The number of links in the component G' is denoted by k, where $k \geq m - 1$, otherwise the component would not be connected. For example, a path containing all agents would have $m - 1$ edges. The total utility of the direct links in the component is given by $k(2\delta - 2c)$. There are at most $\frac{m(m-1)}{2} - k$ left over links in the component that are not created yet. The utility of each of these left over links is at most $2\delta^2$ (it has the highest utility if it is in the second-order neighborhood). Therefore, the total utility of the component is at most

$$k2(\delta - c) + \left(\frac{m(m-1)}{2} - k\right)2\delta^2. \tag{8.6}$$

Consider a star with m agents. See as an example a star containing four agents in Fig. 8.2. The star has $m - 1$ agents which are not in the center of the star. The utility of any direct link is $2\delta - 2c$ and of any indirect link $(m - 2)\delta^2$, since any agent is two links away from any other agent (except the center of the star). Thus, the total utility of the star is

[16] Considering two agents playing a game (e.g., trading of knowledge) and each adopting a certain strategy. A Nash equilibrium is characterized by a set of strategies where each strategy is the optimal response to all the others.

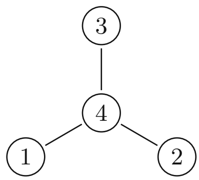

Fig. 8.2 A star encompassing 4 agents

$$\underbrace{(m-1)(2\delta-2c)}_{\text{direct connections}}+\underbrace{(m-1)(m-2)\delta^2}_{\text{indirect connections}}. \tag{8.7}$$

The difference in total utility of the (general) component and the star is just $2(k-(m-1))(\delta-c-\delta^2)$. This is at most 0, since $k \geq m-1$ and $c > \delta-\delta^2$, and less than 0 if $k > m-1$. Thus, the value of the component can equal the value of the star only if $k = m-1$. Any graph with $k = m-1$ edges, which is not a star, must have an indirect connection with a distance longer than 2, and getting a total utility from indirect connections less than $2\delta^2$. Therefore, the total utility of the indirect links will be below $(m-1)(m-2)\delta^2$ (which is the total utility from indirect connections of the star).

If $c < \delta-\delta^2$, then any component of a strongly efficient network must be a star. In a similar fashion, it can be shown (Jackson and Wolinsky, 1996) that a single star of $m+n$ agents has a higher total utility than two separate stars with m and n agents. Accordingly, the star is a strongly efficient network.

3. A star encompassing every agent has a positive value only if $\delta + \frac{n-2}{2}\delta^2 > c$. This is an upper bound for the total achievable utility of any component of the network. Thus, if $\delta + \frac{n-2}{2}\delta^2 < c$ the empty graph is the unique strongly efficient network.

Proposition 4 *In the connections model in which the utility of each agent is given by (8.4), we have*

1. *A pairwise stable network has at most one (non-empty) component.*
2. *For $c < \delta-\delta^2$, the unique pairwise stable network is the complete graph K_n.*
3. *For $\delta-\delta^2 < c < \delta$, a star encompassing every agent is pairwise stable, but not necessarily the unique pairwise stable graph.*
4. *For $\delta < c$, any pairwise stable network that is non-empty is such that each agent has at least two links (and thus is efficient).*

Proof 1. Let us assume, for the sake of contradiction, that G is pairwise stable and has more than one non-empty component. Let u^{ij} denotes the utility of agent i having a link with agent j. Then, $u^{ij} = u_i(G+e_{ij}) - u_i(G)$ if $e_{ij} \notin E(G)$ and $u^{ij} = u_i(G) - u_i(G-e_{ij})$ if $e_{ij} \in E(G)$. We consider now $e_{ij} \in E(G)$. Then $u^{ij} \geq 0$. Let e_{kl} belong to a different component. Since i is already in a component with j, but k is not, it follows that $u^{jk} > u^{ij} \geq 0$, because agent k will receive an additional utility of δ^2 from being indirectly connected to agent i. For similar reasons $u^{jk} > u^{lk} \geq 0$. This means that both agents in the separate component would have an incentive to form a link. This is a contradiction to the assumption of pairwise stability.

2. The net change in utility from creating a link is $\delta - \delta^2 - c$. Before creating the link, the geodesic distance between agent i and agent j is at least 2. When they create a link, they gain δ but they lose the previous utility from being indirectly connected by some path whose length is at least 2. So if $c < \delta - \delta^2$, the net gain from creating a link is always positive. Since any link creation is beneficial (increases the agents' utility), the only pairwise stable network is the complete graph, K_n.
3. We assume that $\delta - \delta^2 < c - \delta$ and show that the star is pairwise stable. The agent in the center of the star has a distance of 1 to all other agents and all other agents are separated by two links from each other. The center agent of the star cannot create a link, since she has already maximum degree. She has no incentive to delete a link either. If she deletes a link, the net gain is $c - \delta$, since there is no path leading to the then disconnected agent. By assumption, $\delta - \delta^2 < c < \delta$, $c - \delta < 0$ and the gain is negative, and the link will not be removed. We consider now an agent that is not the center of the star. She cannot create a link with the center, since they are both already connected. The net gain of creating a link to another agent is $\delta - \delta^2 - c$, which is strictly negative by assumption. So she will not create a link either. The star is pairwise stable.

   The star encompasses all agents. Suppose an agent would not be connected to the star. If the center of the star would create a link to this agent, the net gain would be $\delta - c > 0$ and the benefit of the non-star agent is again $\delta - c > 0$. So both will create the link.

   The star is not the unique pairwise stable network. We will show that for four agents, the cycle, C_4 is also a pairwise stable network. Consider Fig. 8.3.

   If agent 3 removes a link to agent 4, then her net gain is $c - \delta - \delta^3$. For the range of costs of $\delta - \delta^2 < c < \delta - \delta^3 < \delta$, she will never do it. If agent 3 adds a link to agent 1, Fig. 8.4, the net gain is $\delta - \delta^2 < 0$. Thus, for $n = 4$ and $\delta - \delta^2 < c < \delta - \delta^3$, then there are at least two pairwise stable networks: the star and the cycle.
4. For $\delta < c$, the star is not a pairwise stable network because the agent in the center of the star would gain $c - \delta$ from deleting a link. Moreover, it can be

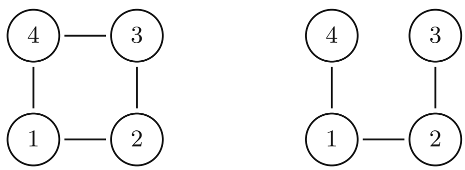

Fig. 8.3 A cycle of four agents (*left*) and the resulting graph (*right*) after the deletion of a link from agent 3 to agent 4

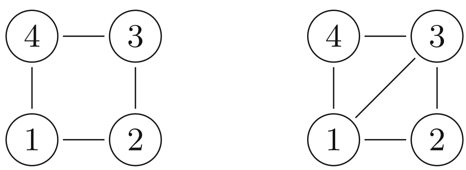

Fig. 8.4 A cycle of four agents (*left*) and the resulting graph (*right*) after the creation of a link from agent 3 to agent 1

shown (Jackson and Wolinsky, 1996) that any connected agent has at least two links.

One can see, from the two propositions described above, that a pairwise stable network is not necessarily efficient. For high cost, i.e., $c > \delta$ there are non-empty pairwise stable networks but they are not efficient.

We now come to the evolution of the network as described in Jackson and Watts (2002). The network changes when agents create or delete a link. At every time step an agent is chosen at random and tries to establish a new link or delete an already existing one. If a link is added, then the two agents involved must both agree to its addition, with at least one of them strictly benefiting (in terms of a higher utility) of the new link. Similarly, a deletion of a link can only be in a mutual agreement. This adding and deleting of links creates a sequence of networks. A sequence of networks created by agents myopically adding and deleting links is called an *improving path*[17] (Jackson and Watts, 2002).

There is a small probability, ϵ, that a mistake occurs (trembling hand) and the link is deleted if present or added if absent. ϵ goes to zero in the long run, $\lim_{t\to\infty} \epsilon(t) = 0$. By introducing this decreasing error ϵ in the agent's decisions, the evolution of the network becomes a Markov process[18] with a unique limiting stationary distribution of networks visited (Jackson and Watts, 2002).

The following definition is important to describe the stochastic evolution of the network.

Definition 5 A network is evolutionary stable if it is in the limiting stationary distribution of networks of the above mentioned Markov process.

We have already investigated the structure and stability of the star, Fig. 8.2, and the cycle, Fig. 8.3. In Jackson and Watts (2002) it is shown that for the case of four agents, the evolutionary stable networks indeed are the stars and cycles. So the network of agents evolves into a quite simple equilibrium configuration.

8.2.3 The Connections Model and Small-World Networks

Carayol and Roux (2005, 2003) propose a model of innovation networks in which networks emerge that show the properties of a "small world."[19] This model is an

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[17] Each network in the sequence of network updates differs in one link from the previous one. An improving path is a finite set of networks $G_1, ..., G_k$ in which one agents is better off by deleting a link (G_{k+1} has one link less than G_k) or two agents are better off by adding a link (G_{k+1} has one link more than G_k).

[18] A Markov process is a random process whose future states are determined by its present state and not on the past states, i.e., it is conditionally independent on the past states given the present state.

[19] A small-world network combines high clustering (high probability that your acquaintances are also acquaintances to each other) with a short characteristic path length (small average distance between two nodes) (Watts and Strogatz, 1998).

extension of the above described connections model, Sect. 8.2.2, and it uses the same notion of pairwise stability and efficiency.

We now give a sketch of the model. Agents are localized on a cycle and benefit from knowledge flows from their direct and indirect neighbors. Knowledge transfer decays along paths longer than one link. This means that less knowledge is received, the longer the path between the not directly connected agents is. The transfer rate is controlled by an exogenous parameter, δ. Each agent has a probability to innovate that is dependent on her amount of knowledge. The knowledge level of an agent is dependent on two factors. (i) the in-house innovative capabilities of the agent and (ii) the knowledge flows coming directly from the neighbors or indirectly (with a certain attenuation factor) from those agents that are connected to the neighbors.

Agent i supports costs, for direct connections which are linearly increasing with geographic distance, that is the distance on the cycle on which they recede. Agent i's utility u_i at a time t is given by the following expression:

$$u_i(G(t)) = \sum_{j \neq i} \delta^{d_{ij}} - c \sum_{j \in N_i} d'_{ij}, \tag{8.8}$$

where d_{ij} is the geodesic distance between agent i and agent j. $\delta \in (0, 1)$ is a knowledge decay parameter and $\delta^{d_{ij}}$ gives the payoffs resulting from the direct or indirect connection between agent i and agent j. c is a positive constant. d'_{ij} describes the geographic distance between agent i and agent j, that is the distance on the cycle. This is the main difference in the assumptions compared to the connections model discussed in Sect. 8.2.2.

Agents are able to modify their connections. This is where the network becomes dynamic. Pairs of agents are randomly selected. If the two selected agents are directly connected they can jointly decide to maintain a link or unilaterally decide to sever the link. If they are not connected, they can jointly decide to form a link. The decision is guided by the selfishness of the agents, which means that they only accept links from which they get a higher utility.

The stochastic process of adding links to the network can be seen as a Markov process where each state is the graph structure at a certain time step. The evolution of the system is a discrete time stochastic process with the state space of all possible graphs. A small random perturbation where the agents make mistakes in taking the optimal decision to form a link or not is introduced. Agents are making errors with a probability $\epsilon(t)$. This error term decreases with time, $\lim_{t\to\infty} \epsilon(t) = 0$.

The introduction of ϵ enables us to find long-run stationary distributions that are independent of initial conditions (the ergodicity of the system) (Jackson and Watts, 2002). Simulations are used in order to find these stationary distributions. Agents are forming and severing links until the network reaches a pairwise stable configuration, where the agents have no incentive to create or delete links any more. The set of stochastically stable networks selected in the long run is affected by the rate of knowledge transfer, δ. The authors find critical values of this parameter for which stable "small-world" networks are dynamically selected (Carayol and Roux, 2005). This is the main difference in the resulting equilibrium network structure to the connections model, in which simpler network configurations are obtained.

8.2.4 Introducing Heterogeneous Knowledge

Ricottilli (2005, 2006) studies the evolution of a network of agents that improve their technological capabilities through interaction while knowledge is heterogeneously distributed among agents. In addition to the sharing of knowledge, each agent is assumed to have an "in-house" innovative capability. Considerable effort is necessary for this "in-house" research and as research is not always successful, it is assumed to change stochastically.

An agent i's innovation capability, V_i, is given by

$$V_i(t) = \sum_{j=1}^{n} a_{ij} b_{ij}(t) V_j(t) + C_i(t) \tag{8.9}$$

with an economy consisting of n agents. a_{ij} =const. is the broadcasting capacity of agent j to agent i and $a_{ii} = 0$ since no agent can broadcast information to herself. The matrix A with elements a_{ij} indicates the total technological information broadcasting capability of this economy. The proximity matrix elements $b_{ij}(t)$ are either 0 or 1 according to whether agent j is identified from agent i as an information supplier. This is the neighborhood of agent i. $C_i(t) \in (0, 1)$ is the in-house capability of agent i. This is a stochastic variable.

Each agent i assesses the value of knowledge of its neighbors (where $b_{ij} \neq 0$), which are the addends of the first term in (8.9). From this function the least contributing one, denoted by $\gamma_i(t)$, is selected.

$$\gamma_i(t) = \min_{1 \le j \le N}\{a_{ij} b_{ij}(t-1) V_j(t-1)\}. \tag{8.10}$$

In a random replacement procedure (search routine) an agent selects either its neighbors and second neighbors (local, weak bounded rationality) or the entire economy excluding its first and second neighbors (global, strong bounded rationality). By doing so, agent i assigns a new member j to the set of information suppliers, setting b_{ij} from 0 to 1. This selection is only accepted if

$$V_i(t) > V_i(t-1). \tag{8.11}$$

The population of agents is classified according to the size of the set of other agents by which they are observed. Global paradigm setters are agents that are observed by almost all agents in the economy. Local paradigm setters are observed by almost all agents belonging to the same component.

Simulations of the evolution of the network show that stable patterns emerge. When the knowledge-heterogeneity of the economy is not very high, global paradigm setters emerge. For high levels of heterogeneity the economy becomes partitioned into two separate halves. In each homogeneous one, local paradigm setters emerge. Ricottilli (2006, 2005) shows that the highest technological capabilities are achieved neither with a local search routine in which only the second neighbors are included nor in global search routines that span the whole economy. Rather, a combination of both improves the system's innovative efficiency the most.

8.2.5 Emerging Cyclic Network Topologies

When studying multi-sector trading economies and input–output systems, Rosenblatt (1957) already identified the importance of circular flows and "feedback" input dependencies between industries (realized by subgraphs called "cyclic nets"). A sufficient condition for a strongly connected network (in which there exits a path from every agent to every other agent and that has an irreducible adjacency matrix) is the existence of a cycle. Subsequent works (Baldry and Ghosal, 2005; Maxfield, 1994) have further incorporated cyclic network topologies (or strong connectivity which implies the existence of a cycle in the network) for the existence of a competitive economy. More recently, Kim and Wong (2007) studied a generalized model of Bala and Goyal (2000) and found that the equilibrium networks consist of cycles (so-called "sub-wheel partitions").

In the following sections, we will focus on some recent network models of knowledge transaction and innovation (the creation of new knowledge) in which cyclic interactions of agents emerge. In Sect. 8.3, we will study a new model of evolving innovation networks. Similarly to the above mentioned authors, we find that the existence of equilibrium networks with a positive knowledge production depends critically on the existence of cycles in the network.

8.2.5.1 Production Recipes and Artifacts

We start by reviewing a model by Lane (2005) in which agents try to produce and sell artifacts. These artifacts can be manufactured according to a production recipe. Such a recipe can either be found independently or through the sharing of knowledge with other agents, which in turn can lead to an innovation, that is the discovery of a new recipe.

Let us denote with r_{ik} the kth recipe of agent i. There is an external environment which consists of external agents (customers) and artifacts which are not produced in the model. At each time period t one agent i is randomly chosen. Then the following steps are taken:

1. The agent tries to get the input required for each recipe r_{ik}. If it is in the agent's own stock then she can produce immediately. If it is not, she buys it from another agent and if it cannot be bought she moves to another recipe.

2. The agent chooses a goal, i.e., the product she wants to produce (one that gives high sales). Therefore she has to find the right recipe for the goal. She produces the product if a successful recipe is found. This can be achieved in two ways. The agent either can try to innovate by herself or she can try to innovate together with another agent.
3. The wealth of agent i at time $t+1$, $w_i(t)$, is calculated according to

$$w_i(t+1) = w_i(t) + \sum_{k=1}^{N_k} n_{ik}(t) - w_i(t) \sum_{l=1}^{N_l} p_{il} c_{il} - \lambda w_i(t), \tag{8.12}$$

   where N_k is the number of products sold, $n_{ik}(t)$ is the number of units sold of product k belonging to agent i, p_{il} is the number of products produced with recipe r_{il}, and c_{il} is the production cost.
   The last term $-\lambda w_i(t)$ guarantees that the wealth of an agent that has not sold any products and does not have any active recipes vanishes.
4. The recipes that could not be successfully used to produce products are canceled.
5. The set of acquaintances of an agent is enlarged. This is possible when two or more agents that have goals which are close in artifact space (i.e., they require similar inputs) cooperate to produce that artifact.
6. With a certain probability dead agents are substituted.

The basic dynamics, absent innovation, is one of production and sales, where the supply of raw materials is external as well as final product demand. There are two main differences to most agent-based innovation models. First, here the agents try to develop new recipes in order to produce products with high sales, as opposed to many agent-based models where the generation of novelty is driven by some stochastic process. Second, in simulations Lane (2005) shows that the network of customers and suppliers often forms closed, self-sustaining cycles.

8.2.5.2 An Autocatalytic Model with Hypercycles

Padgett (1996) and Padgett et al. (2003) introduces an autocatalytic model, based on a hypercycle[20] model. Here agents are represented as skills and these skills are combined in order to produce. Skills, like chemical reactions, are rules that transform products into other products.

In the following, we will give a short overview of the model.[21] There are two main aspects in the dynamic interactions between the agents: The process of production and the process of learning. The process of production includes three entities: skills, products, and agents. Skills transform products into other products. The skills are features of the agents. On a spatial grid the agents are arrayed with periodic

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[20] A hypercycle is a system which connects self-replicative units through a cycle linkage (Eigen and Schuster, 1979).

[21] This agent-based model is publicly available on the website http://repast.sourceforge.net/examples/index.html under the application module hypercycle.

boundaries. Each agent has eight possible neighbors. At each asynchronous iteration a random skill is chosen. An agent with that skill randomly chooses an input product. If this product fits to the skill then the product is transformed. The transformed product is passed randomly to the neighbors of the agent. If the trading partner has the necessary skill it transforms the product further and passes it on. If the agent doesn't have the compatible skill, the product is ejected into the output environment and a new input product is selected.

One can look at the production process from a wider perspective. An input product comes from the environment, then passes through production chains of skills until it is passed back as output to the environment. These chains self-organize because of a feedback mechanism of the agents. This mechanism is learning through the trade of products.

The process of learning is modeled as learning by doing. If a skill transforms a product and then passes it on to another transforming skill, then the skill is reproduced (learned). Whenever one skill is reproduced anywhere in the system then another one is deleted at random to keep the overall number of skills constant. The agents are able to learn new skills by practicing them and they can forget skills they did not use for a certain period of time. This procedure of learning introduces a feedback mechanism. When an agent loses all its skills, then it is assumed to never recover.

In Padgett (1996) and Padgett et al. (2003) the emergence of self-reinforcing hypercycle production chains is shown. In these hypercycles agents reproduce each other through continuous learning. Such cycles generate a positive growth effect on the reproduction of skills. Thus, even in a competitive environment the sharing of knowledge is crucial to the long-run performance of the system.

8.3 A New Model of Evolving Innovation Networks

8.3.1 Outline of the Modeling Framework

In this section, we study the evolution of networks of agents exchanging knowledge[22] in a novel framework. The network can evolve over time either, by an external selection mechanism that replaces the worst performing agent with a new one or, by a local mechanism, in which agents take decisions on forming or removing a link. In the latter case, we investigate different modalities of interaction between agents, namely bilateral interactions, representing R&D collaborations (Hagedoorn et al., 2006, 2000) or informal knowledge trading (Von Hippel, 1987), versus unilateral interactions (similar to Bala and Goyal (2000) agents decide unilaterally whom to connect to), representing a generalization of informal knowledge trading. We further study the impact of varying costs for maintaining links and the

[22] See also the chapter of Robin Cowan and Nicolas Jonard in this book as well as Cowan and Jonard (2004), Cowan et al. (2004).

impact of augmenting or diminishing effects on the value of knowledge with the number of users associated with different types of knowledge. Our model exhibits equilibrium networks and we compare their structure and performance. Similar to the models discussed in the last section we will show that cyclic patterns in the interactions between agents play an important role for the stability (permanence) and performance of the system.

We study different assumptions on the behavior of agents. In the most simple case, denoted by *Extremal Dynamics*, agents form links at random and, through an external market selection mechanism, the worst performing agent (this is where the denotation extremal stems from) is replaced with a new one. In this setting, agents are completely passive and they are exposed to a least-fit selection mechanism.

In a more realistic setting, called *Utility Driven Dynamics*, agents choose with whom to interact, but their behavior is still boundedly rational and does not consider strategic interaction. The way in which agents create or delete links to other agents is a trial and error process for finding the right partner. Here we study two different modes of interaction. In the first interaction mode, agents are creating bilateral links. Bilateral links represent formal R&D collaborations among agents (Hagedoorn et al., 2000), or informal knowledge trading (Von Hippel, 1987). In the second interaction mode, agents are transferring knowledge unilaterally, which means that one agent may transfer her knowledge to another but the reverse is not mandatory. In this setting, the transfer of knowledge may be reciprocated, but knowledge can also be returned from a third party. In the latter case, we speak of indirect reciprocity. If knowledge is transferred unilaterally, the innovation network can be represented as a directed graph comprising unilateral links, while if all interactions are bilateral, the innovation network can be represented as an undirected graph.

In the setting of unilateral links, we also investigate the impact of additional benefits from network externalities. These benefits consider specific structural properties of networks which have an augmenting effect on the value of knowledge. We study two different types of network properties which increase the value of knowledge. We call these types *Positive Network Externalities*. The first Positive Network Externality considers the factor that the more the centrality of an agent rises with the creation of a link, the higher is the benefit from that link. A high centrality indicates that an agent is connected to other agents through short paths. This means that, when knowledge travels along short distances between agents, it has a higher value than knowledge that has to be passed on between many agents. This effect can be captured by introducing an attenuation of knowledge with the distance it has to travel (by getting passed on from one agent to the next) until reaching an agent. The second Positive Network Externality captures an opposite effect when knowledge is passed on from one agent to the next. Here the value of knowledge increases with the number of transmitters (who are also user) of that knowledge. More precisely, we assume that feedback loops create an increase in the value of knowledge of the agents that are part of the loop. The more the agents absorb and pass on knowledge the higher is the value of that knowledge. This means that a link that is part of a long feedback loop increases the value of the knowledge passed on from one agent to the next.[23]

We can summarize the different settings that are studied in this section as follows. We investigate the performance and evolution under the two aforementioned assumptions on the behavior of agents, namely Extremal Dynamics and Utility Driven Dynamics. In the latter setting, we further study the effect of different modes of interaction, i.e., bilateral and unilateral knowledge transactions among agents. When studying unilateral interaction among agents, we introduce different augmenting processes on the value of knowledge depending on the structure of the network, called Positive Network Externality. We study the impact of an attenuation of the value of knowledge by the distance from the giver to the receiver as well as the contrary effect of an increase of the value of knowledge with the number of users of that knowledge depending on the type of knowledge under investigation. Finally, we discuss the networks obtained under these different settings with respect to their topologies and performance.

8.3.2 Bilateral Versus Unilateral Knowledge Exchange

We interpret bilateral interactions as R&D collaborations on a formal or informal basis (Hagedoorn et al., 2000). Both parties involved share their knowledge in a reciprocal way, that means one agent is giving knowledge to another if and only if the other agent is doing this as well and both agents benefit from this transaction.

We then compare bilateral interaction with the case of agents sharing knowledge in a unidirectional way with other agents. They then maintain only those interactions that are in some form reciprocated (and this way lead to an increase in their knowledge levels after a certain time) but not necessarily from the agent they initially gave their knowledge to (indirect reciprocity). The latter is referred to unilateral knowledge exchange which can be seen as a generalization of informal knowledge trading.

In the case of informal knowledge trading, agents exchange knowledge if both strictly benefit. Instead, in the case of generalized informal knowledge trading, one agent transfers knowledge to another one without immediately getting something back. After a certain time (time horizon T) an agent evaluates its investment by assessing its total net increase in knowledge. By introducing unilateral knowledge exchange we relax two requirements: (i) we do not require that the investment in sharing ones knowledge has to be reciprocated instantaneously and in a mutually concerted way. And (ii) the reciprocation does not necessarily have to come from the same agent. With this generalization we introduce that (i) agents have only limited information on the value of knowledge of others and on the network of interactions. (ii) Agents proceed in a trial and error fashion to find the right partners

[23] We study closed loops, because we assume that knowledge issued from one agent has to return to that agent in order for her to take advantage of this added value of knowledge (created by the multiplicity of other users).

for exchanging their knowledge. In this setting reciprocity emerges either directly or indirectly.

If the total knowledge level of an agent at the time horizon T is higher than it was when the agent started to share her knowledge with another agent, this interaction is evaluated beneficial, otherwise it is not. Only if the interaction is evaluated beneficial, the agent continues sharing its knowledge with the other agent, otherwise it stops the interaction. This procedure requires only limited information on the other agents, since the agent cares for its own total increase in knowledge and does not need to evaluate the individual knowledge levels of others. We describe this link formation mechanism in more detail in Sect. 8.3.9.3.

8.3.3 Unilateral Knowledge Exchange and Reciprocity

If the interaction of agents are unilateral then agents invest into innovation by sharing knowledge with other agents. An investment is an advance payment with the expectation to earn future profits. When one agent transfers knowledge to another one without immediately getting something back, this can be regarded as an investment. There are usually two ways in which an investment can be expected to bring in reasonable returns.

One way is the creation of contracts. As a precondition for contracts technologies must be protectable by intellectual property rights (IPR). Otherwise agents cannot trade them (once the technology is offered, i.e., made public, everybody can simply copy it and there is no more need to pay for it). Contracts must be binding and complete (Dickhaut and Rustichini, 2001). The contract has to be binding or agents may not meet their agreement after the payment has been made. It has to be complete, or uncertain agreements may lead agents to interpret it in a way most favorable to their position and this can cause agents to retreat from the contract.

The requirements for contracts can be difficult to realize. Another way is to expect reciprocative behavior to the investment. The beneficiary can either directly or indirectly reciprocate the benefit. Direct reciprocity means to respond in kind to the investor, and indirect reciprocity to reward someone else than the original investor.

One of the possible explanations for reciprocal behavior (Bolton and Ockenfels, 2000; Fehr and Fischbacher, 2003; Fehr and Schmidt, 1999; Nowak and Sigmund, 1998, 2005) (see, e.g., Dieckmann, 2004, for a survey) is to assume the existence of reputation. Agents believe that if they invest into another agent they will increase their reputation and then realize a reasonable return coming back to them directly or indirectly ("strategic reputation building").

In reality, only partial information about reputation is available and experimental works show that, even in the absence of reputation, there is a non-negligible amount of reciprocal cooperative behavior among humans (Bolton et al., 2005). As Dickhaut and Rustichini (2001) put it, "...investment occurs even though agents cannot create binding contracts nor create reputation." Thus, agents invest into each other by transferring their knowledge even if they cannot immediately evaluate the benefit from this investment.

We assume that agents are not a priori reciprocating if they receive knowledge from others. But they perceive that interactions that are reciprocated in some way are beneficial (increasing their own knowledge) and these are the interactions that they maintain in the long run.

The problems associated with bilateral exchange of knowledge (direct reciprocity) and experimental evidence suggest that unilateral knowledge exchange, in which indirect reciprocity can emerge, is a relevant mode of interaction between agents. Moreover, the fact that interactions between anonymous partners become increasingly frequent in global markets and tend to replace the traditional long-lasting mutual business relationships poses a challenge to economic theory and is one of the reason for the growing interest about indirect reciprocity in the economic literature.

8.3.4 Indirect Reciprocity, Directed Graphs, and Cycles

An R&D network can be described as a graph in which agents are represented by nodes, and their interactions by directed links. Indeed, as mentioned above, if agent i transfers knowledge to agent j (e.g., by providing a new technology), the reverse process, i.e., that agent j in turn transfers knowledge to i, is in principle not mandatory. This means that the links representing the transfer of knowledge are directed. The underlying graph can be represented by an adjacency matrix, $\mathbf{A}$ with elements $a_{ij} \in [0, 1]$, which is not symmetric, $a_{ij} \neq a_{ji}$. In other words, directed means that we distinguish the pairs (i, j) and (j, i) representing the links from i to j and from j to i, respectively. On the other hand, if the adjacency matrix is symmetric, it means that any two agents are connected both by a link from i to j and by a link from j to i. We say, in this case, that they are connected by a *bidirectional* link. Notice that the symmetry also implies that the two links have identical weights.

Reciprocity requires the presence of cycles. In particular, direct reciprocity corresponds to a cycle of order $k = 2$, while indirect reciprocity corresponds to a cycle of order $k \geq 3$ (see Figs. 8.5 and 8.6). Therefore, the emergence and permanence of

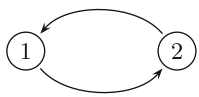

Fig. 8.5 A cycle of length 2 represents an interaction between agents that is direct reciprocal

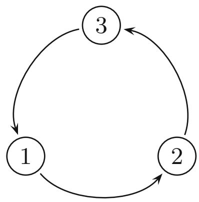

Fig. 8.6 A cycle of length 3 (or longer) represents an interaction between agents that is indirect reciprocal

direct/indirect reciprocity is deeply connected to the existence of cycles and in the graph of interactions.

8.3.5 Formal Modeling Framework

In this section, we formalize the general framework for the investigation of evolving networks of selfish agents engaged in knowledge production via the sharing of knowledge. In such a framework, it is possible to investigate how the emergence and permanence of different structures in the network is affected by (1) the form of the **growth function** of the value of knowledge, (2) the length of **time horizon** after which interactions are evaluated and (3) the **link formation/deletion rules**. At a first glance, this problem includes a multitude of dimensions, as the space of utility functions and link formation/deletion rules is infinite. However, some natural constraints limit considerably the number of possibilities and make a systematic study possible. In the following, we present the general framework. We then focus on a subset of the space of utility functions and link formation rules. For these, we present briefly some analytical results, but since the value of knowledge of an agent is assumed to be a non-linear function of the neighboring agents, we illustrate them in terms of computer simulations. We finally summarize the results and discuss them in relation to the context of innovation.

We consider a set of agents, $N = \{1, ..., n\}$, represented as nodes of a network G, with an associated variable x_i representing the value of knowledge of agent i. The value of knowledge is measured in the units of profits an agent can make in a knowledge-intensive market. It has been shown that the growth of such knowledge-intensive industries is highly dependent on the number and intensity of strategic alliances in R&D networks (Powell and Grodal, 2006). In our model we bring the value of knowledge of an agent, denoted by $x_i(t)$, at time t in relation with the values of knowledge of the other agents $x_j(t)$ at time t in the economy, that are connected to the current agent i. A link from i to j, e_{ij}, takes into account that agent i transfers knowledge to agent j. The idea is, that through interaction, agents transfer knowledge to each other which in turn increases their values of knowledge.

We focus here only on the network effects on the value of knowledge of an agent. We therefore neglect the efforts of agents made to innovate on their own, without the interaction with others.[24] In particular, we assume that the growth of the value of knowledge of agent i depends only on the value of knowledge of the agents, j, with outgoing links pointing to him (those who transfer knowledge to her), $j \in V(G)$ such that $e_{ji} \in E(G)$.

[24] The "in-house" R&D capabilities of an agent could be introduced by an additional (stochastic) term $S_i(x_i)$. Similar to Ricottilli (2006) in Sect. 8.2.5, $S_i(x_i)$ captures the innovation activities of agent i without the interaction with other agents. We assume that the "in-house" capabilities of agents are negligible compared to network effects. Thus, we concentrate only on network effects on the increase or decrease in the value of knowledge.

In a recent study on the dynamics of R&D collaboration networks in the US IT industry, Hanaki et al. (2007) have shown that firms form R&D collaborations in order to maximize their net knowledge (information) flow. Cassiman and Veugelers (2002) suggested that this knowledge flow can be decomposed into incoming and outgoing spillovers capturing the positive and negative effects of R&D collaborations.

We try to incorporate these positive and negative effects into a differential equation that describes the change (increase or decrease) in the value of knowledge of an agent through R&D collaborations with other agents. We assume that the knowledge growth function can be decomposed into a decay term, a benefit term, and a cost term depending on the interactions of an agent. The equation for knowledge growth reads

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -D_i(x_i) + B_i(\mathbf{A}, \mathbf{x}) - C_i(\mathbf{A}, \mathbf{x}), \tag{8.13}$$

where

$\dot{x}_i$ . . . growth of the value of knowledge of agent i
$\mathbf{A}$ . . . adjacency matrix (representing the network)
$\mathbf{x}$ . . . vector of agents' values of knowledge
$D_i(\dot{x}_i)$ . . . knowledge decay (obsolescence of knowledge)
$B_i(\mathbf{A}, \mathbf{x})$ . . . interaction benefits of agent i
$C_i(\mathbf{A}, \mathbf{x})$ . . . interaction costs of agent i

$\mathbf{B} \geq 0$ and $\mathbf{C} \geq 0$ are benefit and cost terms, respectively, while $\mathbf{D} \geq 0$ is a decay term which includes the fact that a technology loses its value over time (obsolescence). In our setting, only through R&D collaborations with other agents, an agent can overcome the obsolescence of knowledge. This ensures that agents who do not interact with others have necessarily vanishing value of knowledge in our model (since $B_i = C_i = 0 \Leftrightarrow a_{ij} = 0\, \forall j$ and thus $\dot{x}_i < 0$). In other words, we investigate an R&D intensive economy in which an agent's performance is critically depending on its R&D collaborations.

Interaction is described by the adjacency matrix $\mathbf{A}$ that contains the elements a_{ij} in terms of 0 and 1. This dynamics can be interpreted as a *catalytic network* of R&D interactions (passing a technology to another agent, R&D collaborations), where the different agents are represented by nodes, and their interaction by links between these nodes, cf. Fig. 8.1. More precisely,

$$a_{ij} = \begin{cases} 1 & \text{if agents } i \text{ transfers knowledge to agent } j \\ 0 & \text{otherwise} \end{cases} \tag{8.14}$$

We noted already that the network of interactions is modeled on a directed graph, which means that the adjacency matrix is not generally symmetric: $a_{ij} \neq a_{ji}$.

The benefit term, $B_i(\mathbf{x}, \mathbf{A})$, accounts for the fact that an agent's value of knowledge increases by receiving knowledge from other agents. The cost term, $C_i(\mathbf{x}, \mathbf{A})$, accounts for the fact that transferring knowledge to other agents is costly. Such a cost can vary in magnitude depending on the technological domain, but, in general, to make someone else proficient in whatever new technology requires a non-null effort.

In the following, we will further specify the growth of the value of knowledge in (8.13). We will make simple assumptions on benefits, $B_i(\mathbf{x}, \mathbf{A})$, and costs, $C_i(\mathbf{x}, \mathbf{A})$, which allow us to derive some analytical results and thus gain some insight on the behavior of the system.

8.3.6 Pairwise Decomposition

Networks are sets of pairwise relationships. In systems of interacting units in physics, a superposition principle holds, such that the force perceived by a unit is due to the sum of pairwise interactions with other units. Similarly, one could think of decomposing both benefits and costs of each agent i in a sum of terms related to the agents j interacting with i. However, this would imply to ignore network externalities[25] (it is very important to note this fact). We will see in the following that externality does play an important role. So far, in the literature on complex networks one has considered only the pairwise interaction term, while the literature on economic networks has focused on some simple externalities such as the network size, or the distance from other agents, see Sect. 8.2.5.

Our approach is to assume that benefit and cost are each decomposable in two terms: one term related to the direct interaction, further decomposable in pairwise terms, and another term related to externality (corresponding to positive and negative externality):

$$B_i(\mathbf{A}, \mathbf{x}) = \sum_j b_{ji}(x_j, a_{ji}) + b^e_{ji}(x_j, \mathbf{A}), \tag{8.15}$$

$$C_i(\mathbf{A}, \mathbf{x}) = \sum_j c_{ij}(x_i, a_{ij}) + c^e_{ij}(x_i, \mathbf{A}, \mathbf{x}), \tag{8.16}$$

where b stands for benefit, c for cost, e for externality. The effect of network externalities will be explained in Sect. 8.3.9.9. Benefit, $b_{ji}(x_j, a_{ji})$, and cost, $c_{ij}(x_i, a_{ij})$, terms are monotonically increasing with the value of knowledge, x_i . They have the following properties:

[25] In our model we define a network externality as a function of the network that affects the utility of an agent.

$$b_{ji}(x_j, a_{ji}) = \begin{cases} 0 & \text{if } a_{ji} = 0 \lor x_j = 0 \\ > 0 & \text{if } a_{ji} = 1 \land x_j > 0 \end{cases}, \tag{8.17}$$

$$c_{ij}(x_i, a_{ij}) = \begin{cases} 0 & \text{if } a_{ij} = 0 \lor x_i = 0 \\ > 0 & \text{if } a_{ij} = 1 \land x_i > 0. \end{cases} \tag{8.18}$$

We assume that benefits are linear functions of the value of knowledge of agent i which shares its knowledge with agent j. We introduce the linear assumption $b_{ji}(x_j, a_{ji}) = a_{ji} x_j$.

In the most simple case costs for transferring knowledge can be neglected, $c_{ij}(x_i, a_{ij}) = 0$. This means that knowledge is fully codified (Foray, 2004) and it can be transferred to another agent without any losses. Further, null costs imply that knowledge is non-rivalrous, meaning that the value of knowledge is not reduced by the use of that knowledge by another agent. When costs are neglected, the growth in the value of knowledge of agent i is given by the following equation (the case of **Null Interaction Costs**, further analyzed in Sect. 8.3.8).

$$\frac{dx_i}{dt} = -dx_i + b \sum_{j=1}^{n} a_{ji} x_j. \tag{8.19}$$

In more realistic setting, costs cannot be neglected. In order to come up with a reasonable expression for these costs, we make some further assumptions. We assume that the higher the value of knowledge of an agent is, the more complex it is. Moreover, the more the complex knowledge is, the more difficult it is to transfer it (Rivkin, 2000; Sorenson et al., 2006). The coordination and processing capabilities of agents are constrained ("managerial breakdown"). Thus, the more complex knowledge gets the higher are the costs for transferring it. The cost, $c_{ij}(x_i, a_{ij})$, for transferring knowledge from agent i to agent j is an increasing function of the value of the knowledge that is to be transferred, x_i. We assume that costs increase by more than a proportional change in the value of knowledge that is being transferred.

$$c_{ij}(\alpha x_i) > \alpha c_{ij}(x_i). \tag{8.20}$$

This characteristic is closely related to decreasing returns to scale and convex cost functions.[26] The most simple setting for such a function is a quadratic term of the form $c_{ij}(x_i, a_{ij}) = c a_{ij} x_i^2$. The growth in the value of knowledge of agent i is then governed by the following equation (the case of **Increasing Interaction Costs**, further analyzed in Sect. 8.3.8):

[26] In the standard economic theory of the agent the extent to which a given input can increase output is usually assumed to be a decreasing function of the input. The output increases at a decreasing rate when the input in production increases (Hausman, 2003).

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\mathrm{d}x_i + b\sum_{j=1}^{n} a_{ji}x_j - c\sum_{j=1}^{n} a_{ij}x_i^2. \tag{8.21}$$

This is an ordinary differential equation with a linear decay, a linear benefit, and quadratic costs.

Equation (8.21) can be interpreted as an extension of a logistic equation. In a complete graph every agent shares her knowledge with every other agent. Starting with the same initial values, this symmetry implies that all knowledge values are identical, i.e., $x_i = x$, equation (8.21) then becomes

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -dx + b(n-1)x - c(n-1)x^2 \\ &\xrightarrow{\frac{d}{b}\ll n} b(n-1)x\left(1 - \tfrac{c}{b}x\right)\end{aligned}. \tag{8.22}$$

Equation (8.22) is similar to the logistic function $\dot{x} = \alpha x(1 - \frac{x}{\beta})$ with parameters $\alpha = b(n-1)$ and $\beta = b/c$.

In the following section, we relate the topology (cyclic topologies in particular) of the network with the long-run values of knowledge of the agents.

8.3.7 Non-permanence of Directed Acyclic Graphs

The study of the relation between the performance of an economy and the underlying network of interactions has already a long tradition, see, e.g., Rosenblatt (1957) ("cyclic nets"). More recently Maxfield (1994) has shown that the existence of a competitive equilibrium is related to the strong connectedness of the network of relations between users and producers in a market economy. Strong connectedness means that there exists a closed walk or a cycle in the network. On the other hand, if there does not exist such a cycle, then the network is not strongly connected. In a similar way in our model strong connectedness is critically influencing the performance of the agents. In the main result of this Sect. 8.11, we show that in our model all values of knowledge vanish if the underlying network of interactions does not contain a cycle.

For the general equation (8.13) we can identify the topology of the network in which agents cannot be permanent. Hofbauer and Sigmund (1998) give the following definition of permanence:

Definition 6 A dynamical system is said to be permanent if there exists a $\delta > 0$ such that $x_i(0) > 0$ for $i = 1, \ldots, n$ implies $\lim_{t\to\infty} \inf x_i(t) > \delta$.

We generalize the above notion of permanence to networks in which the nodes have a state variable attached (that depends on the state variable of their neighbors). If the state variables are non-zero the network is said to be permanent, otherwise it is not. This is justified, since nodes with vanishing state variables have no interactions at all.

First, we have to introduce the definition of graphs which do not contain any closed walks or cycles.

Definition 7 A directed acyclic graph is a directed graph with no directed cycles.

More general, if a graph is a directed acyclic graph then it does not contain a closed walk.

For several proofs in this section we need the following lemma (denoted by the *comparison principle* (Khalil, 1995)).

Lemma 8 *If we consider two time-dependent variables, $x(t)$ and $y(t)$ with different growth functions $g(x)$ and $f(x)$ (continuous, differentiable)*

$$\dot{x} = f(x) \tag{8.23}$$

$$\dot{y} = g(x) \tag{8.24}$$

$$x(0) = y(0) \tag{8.25}$$

and $g(x) \geq f(x)$ then it follows that $y(t) \geq x(t)$. Similarly, if $g(x) \leq f(x)$ then $y(t) \leq x(t)$.

Proof Using Cauchy's mean value theorem for the two continuous, differentiable functions, $x(t)$ and $y(t)$, we have

$$\frac{x'(\tau)}{y'(\tau)} = \frac{x(t) - x_0}{y(t) - y_0} \geq 1 \tag{8.26}$$

with $\tau \in (0, t)$. The inequality holds since $x'(\tau) = f(x(\tau)) \geq y'(\tau) = g(y(\tau))$ $\forall \tau \in (0, t)$. It follows that

$$x(t) - x_0 \geq y(t) - y_0 \tag{8.27}$$

$$x_0 = y_0 \tag{8.28}$$

and thus $x(t) \geq y(t)$. □

If a network is a directed acyclic graph then it does not contain a closed walk. For a directed acyclic graph we can make the following observation.

Proposition 9 *In every directed acyclic graph, there is at least one node v with no incoming links, i.e., a source.*

Proof (Godsil and Royle, 2001) We give a proof by contradiction. We assume that every node has an incoming link. We start with some node u and find an incoming link (x, u) – by assumption every node has at least one incoming link. We go to the destination of the link, x. Again, we can find an incoming link (y, x). We then proceed to node y. There is an incoming link (z, y). We consider node z. After at most $n + 1$ steps, we will visit some node in the graph twice. This is a contradiction to the assumption that the graph is acyclic. □

We can partition the nodes in the network into specific sets which take into account from which other nodes there exists an incoming path to these nodes. We will show that this is important to obtain a result on the permanence of the values of knowledge of the agents.

Definition 10 We denote the set of sources of a directed acyclic graph G by S_0. We say that S_0 is the 0-th order sources of G. The nodes that have only incoming links from S_0 are denoted by S_1, the 1-st order sources of G. We consider the graph $G\backslash S_0$. The nodes that have only incoming links from S_1 in $G\backslash S_0$ (obtained by removing the nodes in S_0 and their incident links from G) are denoted by S_2. Accordingly, the nodes having only incoming links from S_{k-1} in the graph $G\backslash(S_{k-2} \cup \ldots \cup S_0)$ are denoted by S_k, the k-th order sources of G, where $k \leq n$.

We can have at most n such sets in the graph G with n nodes. In this case G is a directed path P_k. Moreover, we have that

Proposition 11 *The nodes in a directed acyclic graph G can be partitioned in the sets S_0, S_1, ..., S_k, $k \leq n$ defined in (8.10).*

Proof From Proposition (9) we know that the directed acyclic graph G has at least one source node. All the sources form the set S_0. If we remove the nodes in S_0 (as well as their incident links) from G then we obtain again a directed acyclic graph $G_1 := G\backslash S_0$ (since the removal of links cannot create cycles). Therefore, Proposition (9) also holds for G_1. We consider the source nodes in G_1. These nodes have not been sources in G and they have become sources by removing the incident links of the sources in G. Thus, the source nodes in G_1 have only incoming links from nodes in S_0. Further on, the sources in G_1 form the set S_1. We can now remove the nodes S_1 from G_1 and obtain the graph G_2 with new sources S_2. We can consider the k-th removal of source nodes. We make the induction hypothesis that the sources of G_{k-1} form the set S_{k-1}. Removing the sources from G_{k-1} gives a directed acyclic graph G_k which contains the sources S_k. One can continue this procedure until all nodes have been put into sets $S_0, S_1, \ldots, S_k$ with at most $k = n$ sets. □

There exists a relationship between the set (defined in (8.8)) a node belongs to and the nodes from which there exists an incoming path to that node.

Corollary 12 *Consider a node $i \in S_j$. Then there does not exist a path from nodes $k \in S_m$, $m \geq j$, to node i. Conversely, node i has only incoming path from nodes in the sets S_0, ..., S_{j-1}.*

Proof Assume for contradiction that there exists such a path from a node $k \in S_m$, $m \geq j$ to a node $i \in S_j$. By the construction of the sets S_j (8.8), node i must be a source with no incoming links after the removal of the sets $S_0, \ldots, S_{j-1}$ from G. But this is a contradiction to the assumption that node j has an incoming link from a node $k \in S_m$, $m \geq j$. □

From the above definition and observations we can derive an upper bound on the values of knowledge of the nodes in a directed acyclic graph.

Proposition 13 *Consider (8.13) with a linear decay $D_i(x_i) = dx_i$, a linear benefit $B_i(\mathbf{A}(G), \mathbf{x}) = b \sum_{j \in N_i^-} x_j$ and a non-negative cost $C_i(\mathbf{A}(G), \mathbf{x}) \geq 0$ where $d \geq 0$, $b \geq 0$. Then for every node i in G there exists a $k \leq n$ such that*

$$x_i(t) \leq (a_k t^k + a_{k-1} t^{k-1} + ... + a_0)e^{-dt}. \tag{8.29}$$

Proof From Proposition (11) we know that the directed acyclic graph G has a partition of nodes into sources $S_0, \ldots, S_k$, $k \leq n$. Consider a node $x_0 \in S_0$. With (8.13) the time evolution of her value of knowledge is given by

$$\dot{x}_0 = -dx_0 - C_0 \leq -dx_0. \tag{8.30}$$

Here we use the fact that $C_0 \geq 0$. The function solving the equation $\dot{x} = -dx$ is an upper bound for $x_0(t)$ (with identical initial conditions), see (8.1).

From Proposition (9) we know that there are first-order sources S_1 in G that have only incoming links from nodes in S_0. The evolution of the value of knowledge for a node $x_1 \in S_1$ is given by

$$\dot{x}_1 = -dx_1 + \sum_{j \in S_0} x_j - C_1. \tag{8.31}$$

The second term on the right-hand side of the above equation contains the sum of all values of knowledge of all nodes in S_0. We know that they are bounded from above by $x(t) \leq x(0)e^{-dt}$. Thus, (8.31) has an upper bound

$$\dot{x}_1 \leq -dx_1 + a_1 e^{-dt} \tag{8.32}$$

with an appropriate constant a_1. The solution of the equation $\dot{x} = -dx + a_1 e^{-dt}$ is given by $x(t) = (a_1 + a_0 t)e^{-dt}$. It follows that

$$x_1 \leq (a_1 + a_0 t)e^{-dt}. \tag{8.33}$$

In the following, we make a strong induction. We have the induction hypothesis that for the $(k-1)$-th order sources there exists an upper bound

$$\dot{x}_{k-1} \leq (a_{k-1} t^{k-1} + a_{k-2} t^{k-2} + \ldots + a_0)e^{-dt} \tag{8.34}$$

and this holds also for all nodes in the sets of sources with order less than $k-1$. We consider the nodes in S_k with $l \in S_k$. We have that

$$\dot{x}_l(t) = -dx_l + b \sum_{j \in N_l^-} x_j - C_l, \tag{8.35}$$

where the in-neighborhood N_l^- contains only nodes in the sets $S_0, \ldots, S_{k-1}$. For these nodes an upper bound is given by (8.34), and thus we get an upper bound for (8.35)

$$\dot{x}_l(t) \leq -dx_l + (a_{k-1}t^{k-1} + a_{k-2}t^{k-2} + \ldots + a_0)e^{-dt}. \tag{8.36}$$

We can now use the following lemma:

Lemma 14 *For an ordinary differential equation of the form*

$$\dot{y} + dy = (a_k t^k + a_{k-1}t^{k-1} + ... + a_2 t + a_1)e^{-dt} \tag{8.37}$$

there exists a solution of the form

$$y(t) = \left(\frac{a_k}{k+1}t^{k+1} + ... + a_0\right)e^{-dt} \tag{8.38}$$

with the limit $\lim_{t\to\infty} y(t) = 0$

Solving for the upper bound from above gives the desired result.

$$x_l(t) \leq (a_k t^k + a_{k-1}t^{k-1} + ... + a_0)e^{-dt}. \tag{8.39}$$

□

With the last Proposition (13) it is straightforward to obtain the following proposition, which is the main result of this section.

Proposition 15 *Consider* (8.13) *with a linear decay* $D_i(x_i) = dx_i$, *a linear benefit* $B_i(\mathbf{A}(G), \mathbf{x}) = b \sum_{j \in N_i^-} x_j$, *and a non-negative cost* $C_i(\mathbf{A}(G), \mathbf{x}) \geq 0$ *where* $d \geq 0$, $b \geq 0$. *If the network* G *is a directed acyclic graph then the values of knowledge vanish. This means that* G *is not permanent.*[27]

Proof From Proposition (13) we know that each node k in the graph G has a value of knowledge which is bounded by $x_k(t) \leq (a_k t^k + a_{k-1}t^{k-1} + \cdots + a_0)e^{-dt}$ for some finite $k \leq n$. Since any finite polynomial grows less than an exponential function we have that $\lim_{t\to\infty} x_k(t) = 0$. This holds for all nodes in G. This completes the proof that for all $i = 1, \ldots, n$ in a directed acyclic graph G we have that $\lim_{t\to\infty} x_i(t) = 0$ and therefore G is not permanent. □

Thus, if agents are permanent, the graph contains a closed walk (or a cycle). If agents get their links attached at random, only those survive, who are part of a cycle. If agents can chose, whom to transfer their knowledge to, then they have to form

[27] Remember that the definition of permanence in (8.6) requires that all nodes have non-vanishing state variables. On the other hand, vanishing state variables would imply that nodes do not interact with each other and the network would not be permanent.

cycles in order to survive. Others (Bala and Goyal, 2000; Kim and Wong, 2007) have found similar results in which the equilibrium network consists of cycles.

There exist a convenient way to identify if a network contains a cycle without actually looking at the permanence of the network which would require to compute the long-run values of knowledge (usually by numerical integration). Instead, from the eigenvalues of the adjacency matrix, $\mathbf{A}(G)$, of a graph, G, one can determine if G contains a cycle. The Perron-Frobenius eigenvalue of a graph G, denoted by $\lambda_{\mathrm{PF}}(G)$, is the largest real eigenvalue of $\mathbf{A}(G)$. The following properties hold (Godsil and Royle, 2001)

Proposition 16 *If a graph G*

1. *has no closed walk, then* $\lambda_{PF}(G) = 0$*,*
2. *has a closed walk, then* $\lambda_{PF}(G) > 1$*.*

Thus, if the graph contains permanent agents, then $\lambda_{\mathrm{PF}}(G) > 1$. Hofbauer and Sigmund (1998), Stadler and Schuster (1996) have found similar conditions under which populations are permanent in a network of replicators.[28]

Finally, we can compute the probability of a network to contain a cycle if links were attached at random.

Proposition 17 *The probability of a random graph* $G(n, p)$ *with n nodes containing a cycle is given by (Jain and Krishna, 2002)*

$$P = \left(1 - (1-p)^{n-1}\right)^n \tag{8.40}$$

which is 0 *if* $p = 0$ *and* 1 *if* $p = 1$.

Proof We can compute the probability of having a closed walk in a random graph $G(n, p)$. Each link is created with probability p. Thus we have a Bernoulli process for the adjacency matrix elements a_{ij} (which indicate if a link exists or not).

$$a_{ij} = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}. \tag{8.41}$$

For every node we have $n - 1$ events to create a link and we are asking for the probability of having at least one of them being created (every node should have at least one incoming link). This is a binomial cumulative function of the form (Durrett, 2004; Casella and Berger, 2001):

$$P = \sum_{k=1}^{n-1} \binom{n}{k} p^k (1-p)^{n-k} \tag{8.42}$$

[28] The replicator equation (in continuous form) is given by: $\dot{x}_i = x_i\left(f_i(\mathbf{x}) - \phi(\mathbf{x})\right)$, where $\phi(\mathbf{x}) = \sum_i x_i f_i(\mathbf{x})$ and $f_i(\mathbf{x})$ is the fitness of species i.

which is equivalent to (8.40), if we use the Binomial theorem

$$(x+y)^n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}. \tag{8.43}$$

□

A similar result to (8.11) has been found by Kim and Wong (2007). The authors study a generalized version of the network formation model introduced by Bala and Goyal (2000).[29] The equilibrium networks in their model are so-called "minimal" graphs, which are graphs that maximize the number of agents that are connected while maintaining only as few links as possible. It is intuitively clear that the most sparse connected graph is a cycle. Thus, the authors find stable equilibrium networks that consist of cycles. However, in Sect. 8.3.9.8 we will show that the network evolution can reduce the set of possible cycles in the equilibrium network such that only the smallest cycles survive.

Thus, cycles play an important role in the evolution of the network and the ability of agents to have non-vanishing knowledge levels. Before we define the evolution of the network in Sect. 8.3.9, we study the dynamics of the values of knowledge for a static network in the next Sect. 8.3.8. There we will further specify the cost functions under investigation: null costs and non-linear costs for maintaining links.

8.3.8 Static Network Analysis

In the following, we analyze the growth functions for the value of knowledge and study two cases separately. In the first, costs are set to zero while in the second costs are a quadratic function of the values of knowledge of the agents.

8.3.8.1 Null Interaction Costs

The most simple case of our general framework is the one of linear benefit and null costs.[30]

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[29] For a further study of Bala and Goyal (2000) applied to information networks see Haller et al. (2007), Haller and Sarangi (2005).

[30] This model has been studied by Jain and Krishna (1998b), Krishna (2003) to explain the origin of life from the perspective of interacting agents. The model of Jain and Krishna intends to describe the catalytic processes in a network of molecular species (which we will denote in the following by agents). However, it was very soon suggested to be applicable to an economic innovation context of interacting agents. In the next sections we will present a more general framework encompassing some of the limitation of the present one. In their model the $\mathbf{x}$ were interpreted as concentrations of chemical species. The a_{ij} are the kinetic coefficients that describe the replication of agents i resulting from *binary* interactions with other agents j.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -dx_i + \sum_{i=1}^{n} a_{ji} x_j. \tag{8.44}$$

In vector notation (8.44) reads:

$$\dot{\mathbf{x}} = (\mathbf{A}^{\mathrm{T}} - d\mathbf{I})\mathbf{x}, \tag{8.45}$$

where $\mathbf{A}^{\mathrm{T}}$ is the transposed of the adjacency matrix and $\mathbf{I}$ is the identity matrix. The solution of the set of equations (8.45) depends on the properties of the matrix $\mathbf{A}$ and has the general form (matrix exponential):

$$\mathbf{x}(t) = e^{-dt} e^{\mathbf{A}^{\mathrm{T}} t} \mathbf{x}(0) \tag{8.46}$$

representing an exponential increase in time of the vector of knowledge values. The relative values of knowledge (shares) are given by

$$y_i = \frac{x_i}{\sum_j x_j}; \quad \sum_j y_j = 1. \tag{8.47}$$

Rewriting (8.44) by means of (8.47) gives us the dynamics of the shares:

$$\dot{y}_i = \sum_j^n a_{ji} y_j - y_i \sum_{k,j}^n a_{jk} y_j. \tag{8.48}$$

Equation (8.48) has the property of preserving the normalization of $\mathbf{y}$. Note that the decay term does not appear in this equation for the relative values. It can be shown (Horn and Johnson, 1990; Boyd, 2006; Krishna, 2003) that the eigenvector to the largest real eigenvalue of $\mathbf{A}^{\mathrm{T}}$ ($\mathbf{A}$ respectively) is the stable fixed point of (8.48).[31] If we consider an eigenvector $\mathbf{y}^{(\lambda)}$ associated with the largest real eigenvalue λ of matrix $\mathbf{A}^{\mathbf{T}}$ (identical to the largest real eigenvalue of $\mathbf{A}$) we have

$$\sum_{j=1}^{n} a_{ji} y_j^{(\lambda)} = \lambda y_i^{(\lambda)}. \tag{8.49}$$

Inserting $\mathbf{y}^{(\lambda)}$ into (8.48) yields

$$\dot{y}_i^{(\lambda)} = \sum_j^n a_{ji} y_j^{(\lambda)} - y_i^{(\lambda)} \sum_{k,j=1}^{n} a_{jk} y_j^{(\lambda)} \tag{8.50}$$

---

[31] If the largest real eigenvalue has multiplicity more than one then the stable fixed point can be written as a linear combination of the associated eigenvector and generalized eigenvectors (Braun, 1993).

$$= \lambda y_i^{(\lambda)} - y_i^{(\lambda)} \underbrace{\sum_{k,j=1}^{n} a_{jk} y_j^{(\lambda)}}_{\lambda \sum_k^n y_k^{(\lambda)} = \lambda} \tag{8.51}$$

$$= \lambda y_i^{(\lambda)} - \lambda y_i^{(\lambda)} = 0. \tag{8.52}$$

Thus, $y_i^{(\lambda)}$ is a stationary solution of (8.48). For the proof of stability see, e.g., Krishna (2003).

8.3.8.2 Increasing Interaction Costs

In the following, we study the evolution of the values of knowledge under a given network structure and we try to compute the fixed points wherever possible. We first show that the values of knowledge are non-negative and bounded. For graphs with two nodes, for regular graphs (including the complete graph), cycles, and stars with an arbitrary number of nodes, we can compute the equilibrium points analytically. For generic graphs with $n \geq 3$ nodes, we have to rely on numerical integrations.

The non-linear (quadratic) dynamical system is given by

$$\dot{x}_i = -dx_i + b \sum_{j=1}^{n} a_{ji} x_j - c \sum_{j=1}^{n} a_{ij} x_i^2 \tag{8.53}$$

with initial conditions, $x_i(0) > 0$. a_{ij} are the elements of the adjacency matrix, $\mathbf{A}$, of a graph G. This can be written as

$$\dot{x}_i = -dx_i + b \sum_{j=1}^{n} a_{ji} x_j - c d_i^+ x_i^2 \tag{8.54}$$

where $d_i^+ = \sum_{j=1}^n a_{ij}$ is the out-degree of node i. In the case of increasing costs we know that the values of knowledge are bounded. We have that

Proposition 18 *For the dynamical system* (8.53) *the values of knowledge are non-negative and finite, i.e.,* $0 \leq x_i < \infty$, $i = 1, \ldots, n$.

Proof For the lower bound $x_i \geq 0$, we observe that

$$\dot{x}_i \geq -dx_i - c(n-1)x_i^2. \tag{8.55}$$

The lower bound is the solution of the equation $\dot{x} = -dx - c(n-1)x^2$. The solution of this equation can be found by solving the corresponding equation for the transformed variable $z = \frac{1}{x}$. We get $x(t) = \frac{de^{da}}{e^{dt} - c(n-1)e^{da}}$ with an appropriate constant $a = \frac{1}{d} \ln \frac{x(0)}{d+(n-1)c}$. Starting from non-negative initial values $x(0) \geq 0$ this lower bound is non-negative as well and approaches null for large t, i.e., $\lim_{t\to\infty} x(t) = 0$. We conclude that $x_i(t) \geq 0$.

In order to compute an upper bound, $x_i \leq$ const. $< \infty$ we first make the following observation. The nodes of a graph, $G = (V, E)$, can be partitioned into nodes without outgoing links, $V_f \subseteq V$ ("free-riders"), and nodes with at least one outgoing link, $V_s \subseteq V$ (sources).

Since the "free-riders" in V_f have no outgoing links, the benefit terms of the sources in V_s are independent of the values of knowledge of the free-riders. Accordingly, a source node $i \in V_s$ has the following knowledge dynamics.

$$\dot{x}_i = -dx_i + b \sum_{j \in V_s \setminus i} a_{ji} x_j - cx_i^2 d_i^+, \tag{8.56}$$

where d_i^+ is the out-degree of node i. We can give an upper bound of

$$\dot{x}_i \leq -dx_i + b \sum_{j \in V_s} x_j - cx_i^2. \tag{8.57}$$

This upper bound has a (finite) fixed point and so does $x_i(t)$. The fixed point is given by

$$dx_i + cx_i^2 = b \sum_{j \in V_s} x_j. \tag{8.58}$$

This is a symmetric equation and therefore all x_i are identical, $x_i = x$. For contradiction assume that there would be $x_i \neq x_j$. Then we have that

$$\underbrace{dx_i + cx_i^2}_{b\sum_{k=1}^n x_k} \neq \underbrace{dx_j + cx_j^2}_{b\sum_{k=1}^n x_k} \tag{8.59}$$

But the left and right side of the equation are identical and so two different x_i, x_j cannot exist.

When all solutions are identical we get $x_i = x = \frac{bn-d}{c}\ \forall i$. Thus, we have shown that there exists an upper bound with a finite fixed point for the source nodes, that is $x_i(t) \leq \infty$, $i \in V_s$.

We now consider the nodes with no outgoing links ("free-riders"). A node $i \in V_f$ follows the dynamics

$$\dot{x}_i = -dx_i + b \sum_{j \in V_s} a_{ji} x_j. \tag{8.60}$$

We have shown already that the source nodes are bounded by some constant, $\sum_{j \in V_s} x_j \leq$ const. Thus, we have that

$$\dot{x}_i \leq -dx_i + \text{const.} \tag{8.61}$$

We have an upper bound of the x_i, $i \in V_f$, given by

$$x_i(t) \leq x_0 e^{-dt} + \frac{\text{const.}}{d} \tag{8.62}$$

with $\lim_{t\to\infty} x_i(t) = \frac{\text{const.}}{d}$. We have shown that for all nodes (sources V_s as well as "free-riders" V_f) $0 \leq x_i < \infty$, $i \in V(G)$. □

For special types of graphs we can deduce further results on the values of knowledge of the agents. First, we can compute the fixed points (given by $\dot{x}_i = 0$) for regular graphs.

Proposition 19 *For any k-regular graph G the fixed point of the values of knowledge is given by $x^* = \frac{kb-d}{kc}$. In particular, the complete graph K_n has the highest total value of knowledge among all regular graphs with $x^* = \frac{(n-1)b-d}{(n-1)c}$.*

Proof The dynamics of the values of knowledge of the nodes in a regular graph with degree $d_i^+ = d_i^- = k$ is given by

$$\dot{x}_i = -dx_i + b \sum_{j \in N_i} x_j - ckx_i^2. \tag{8.63}$$

Starting with homogeneous initial conditions we make the Ansatz $x_i = x$ $i = 1, ..., n$. We get the positive stable fixed points $x^* = \frac{kb-d}{kc}$. □

Second, we can compute the fixed points for cycles.

Proposition 20 *For any cycle C_n the fixed point of the values of knowledge is given by $x^* = \frac{b-d}{c}$.*

Proof The dynamics of the values of knowledge of a cycle C_k of length k is given by

$$\dot{x}_i = -dx_i + bx_{i-1} - ckx_i^2. \tag{8.64}$$

Starting with homogeneous conditions we make the Ansatz $x_i = x$, $i = 1, \ldots, n$. We get the positive stable fixed points $x^* = \frac{b-d}{c}$. □

Third, the fixed points for a star can be computed (the proof can be found in the Appendix (A)).

Proposition 21 *For a star $K_{n,n-1}$ there exists a fixed point which increases with the number of nodes. For $d = 0$ the star has a fixed point of $x^* = \frac{b}{c}$.*

Proof The dynamics of the values of knowledge of a star $K_{1,n-1}$ is given by

$$\dot{x}_1 = +b \sum_{i=2}^{n} x_i - c(n-1)x_i^2, \tag{8.65}$$

$$(\dot{x}_i)_{i>1} = -dx_i + bx_1 - cx_i^2, \tag{8.66}$$

where we assume that all links are bidirectional. Starting with homogeneous initial conditions we make the Ansatz $x_i = x_2$, $i = 2, \ldots, n$. Then x_2 is determined by the root of the polynomial

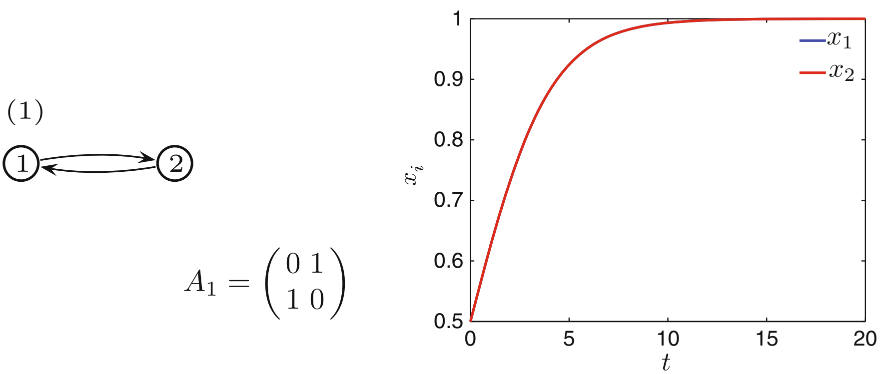
$$x_2^3 + \frac{2d}{c}x_2^2 + \frac{d(cb + (n-1)cd)}{(n-1)c^3}x_2 + \frac{b(d^2 - (n-1)b^2)}{(n-1)c^3} = 0. \tag{8.67}$$

And $x_1 = \frac{d}{b}x_2 + \frac{c}{b}x_2^2$. For $d = 0$, we obtain $x_1^* = x_2^* = \frac{b}{c}$. □

The fixed point increases with the benefit b and decreases with the decay d and the cost c.

We observe that, for vanishing decay, $d = 0$, the fixed point of the system is identical for the regular graph, the cycle, and the star and given by $\frac{b}{c}$. As expected, this fixed point is increasing with the benefit and decreasing with the cost. In a regular graph the fixed point is increasing with the degree k and the asymptotic value (for large k) is $\frac{b}{c}$. Thus, in a regular graph the fixed point ranges for increasing k from $\frac{b-d}{c}$ to $\frac{b}{c}$. Similarly, for the star the fixed point also increases with the number of nodes (i.e., the degree of the central node) but we cannot provide an analytical expression here. On the other hand, the fixed point of the cycle is independent of the length of the cycle. This means that there is no incentive for nodes to be part of larger cycles. And, as we will see in the next section, this limits the growth of the network.

Example 22 We numerically integrate (8.53) for $n = 2$ nodes. We set $d = 0.5$, $c = 0.5$, and $b = 1$. Fixed points are denoted x_i^* for $i = 1, 2$. $x_i^* = 0$ is a fixed point for all graphs.

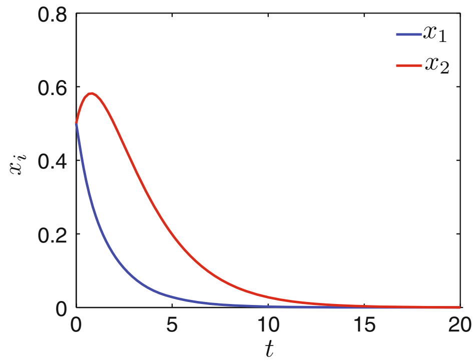


The fixed points are given by

$$x_i^* = \frac{b-d}{c}, \ i = 1, 2, 3. \tag{8.68}$$

(2)

$$A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

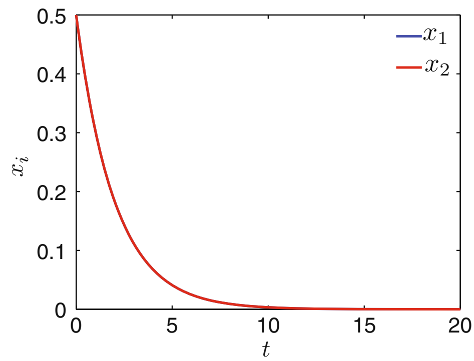


The fixed points are given by

$$x_i^* = 0, \ i = 1, 2, 3. \tag{8.69}$$

(3)

$$A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$



The fixed points are given by

$$x_i^* = 0, \ i = 1, 2, 3. \tag{8.70}$$

In general, the fixed points of (8.53) can only be computed numerically. As an example, we compute the fixed points for all graphs with $n = 3$ nodes for a specific choice of parameters. The results can be found in Appendix (A). In our model we numerically integrate (8.53) for a large time T (and we find that in our simulations the system always reaches a stable fixed point).

8.3.9 Dynamics of Network Evolution

8.3.9.1 Network Evolution as an Iterative Process

After providing the static equilibrium analysis, in this section we turn now to the *dynamics* of the network evolution by investigating different assumptions for the *creation* and *deletion* of links in the network. In particular, we compare two different scenarios, namely the so-called extremal dynamics, where agents do not decide themselves about the link creation and deletion, and the utility driven dynamics, where agents make this decision themselves based on different rules discussed below.

We first define the utility of the agents in our model for a given network G.

Definition 23 Consider a (static) network G. The utility of agent i is given by

$$u_i = \begin{cases} y_i(T), & \text{for Null Interaction Costs} \\ x_i(T), & \text{for Increasing Interaction Costs} \end{cases}, \tag{8.71}$$

where the value of knowledge $x_i(t)$ is given by (8.53) and $\mathbf{A}(G)$, the relative value of knowledge $y_i(t)$ by (8.48) and $\mathbf{A}(G)$. T is called the time horizon.

We assume that the accumulation of knowledge is faster than the frequency of the agents creating or deleting links.[32] With this assumption, we can introduce a *time-scale separation* between the accumulation of knowledge and the evolution of the network.

The evolution of the system is then defined by an alternating sequence of knowledge accumulation, where we keep the network fixed for a given time T, $\mathbf{A}(G)$ =const., and changes in the links (asynchronous updating of the nodes) (see Fig. 8.7). When the knowledge accumulation has reached time T, the network structure is changed. A change in the network takes place by either link addition between two agents i and j, $a_{ij} = 0 \rightarrow a_{ij} = 1$, or by link removal, $a_{ij} = 1 \rightarrow a_{ij} = 0$. When the network has changed, the new utility, determined by (8.18), can be computed for time $2T$. This iterative procedure of knowledge accumulation and link

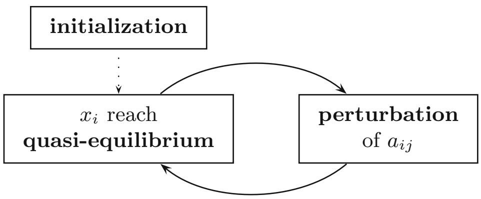


Fig. 8.7 Schematic representation of the network evolution as an iterative process

[32] This means that the value of knowledge on the market (which is not explicitly modeled here) reaches a stationary state determined by the R&D collaborations of each agent (and her neighbors). Only after this adaptation of the evaluation of the stocks of knowledge is finished, i.e., it has reached a stationary state, agents asynchronously change their links.

changes continues for $3T$, $4T$, ... and so on until the network reaches an equilibrium. One can schematically represent this iteration by the following algorithm:

1. initialization: Random graph $G(n, p)$.
2. **quasi-equilibrium**: fast knowledge growth/decline
   With $\mathbf{A}$ fixed, agents evolve according to (8.13) for a given (large) time T.
3. **perturbation**: slow network evolution
   After time T, the network evolves according to two alternative selection processes:

   1. *Extremal Dynamics.*[33]
      The agent with the minimum utility is chosen (if there are more than one agent with the same minimum value, then one of them is chosen at random). The utility of that agent is set to its initial value and all its outgoing and ingoing links are replaced with new random links drawn with probability p from and to all other agents in the system.
   2. *Utility Driven Dynamics*
      An agent is randomly chosen to create or delete one link (unidirectional or bidirectional link formation mechanisms, see Sect. 8.3.9.3). More specifically:

      (i) Either a pair or a single agent is randomly chosen to create or remove a link.
      (ii) The effect of this link decision (creation or deletion) is evaluated at time T. The evaluation can have the following consequences on the link decision.

      - If the utility has increased, then sustain the link decision.
      - If the utility has decreased, then undo the link decision.

4. Stop the evolution, if the network is stable (stability is defined in Sect. 8.3.9.3, otherwise go to **2**

8.3.9.2 Extremal Dynamics Versus Utility Driven Dynamics

Extremal dynamics intends to mimic natural selection (the extinction of the weakest) and the introduction of novelty, which is a global selection mechanism. In contrast, utility driven dynamics is a local selection mechanism that mimics the process by which selfish agents improve their utility through a trial and error process.

The decision upon to add or to remove a link implies a certain level of information processing capabilities (IPC) of the agents. IPC is usually bounded in a complex environment consisting of many other agents and a complex structure of interactions between these agents. In our approach we assume that the agents have no information on the knowledge values of the other agents and only limited information on their links (alliances). They only know with whom they interact directly (their neighborhood). In Table 8.2 we give a short overview of levels of increasing IPC.

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[33] See Bak and Sneppen (1993).

Table 8.2 Increasing levels of agents' information processing capabilities (IPC)

0	Least fit addition/removal of links, e.g Jain and Krishna (1998b)
1	Reactive (passive) acceptance/refusal of link changes.
2	Deliberate decision upon to add/remove a link based on an individual utility function depending on the network, without considering the possible decision of others. An example would be the Connections model discussed in Sect. 8.2.2 with a utility function given by (8.4).
3	Strategic interaction, e.g., Bala and Goyal (2000), considering the possible actions of others

Extremal dynamics refers to a situation in which agents are exposed to link changes that they cannot influence and thus to level 0 in Table 8.2. Utility driven dynamics instead requires a higher level of IPC than a mere acceptance or refusal of link changes. But it requires less IPC than an approach assuming strategic interactions of agents. This follows from the fact that in our model, agents do not estimate how other agents could react on their decisions to change their links. This situation refers to level 2 in Table 8.2. In this chapter, we compare two different settings, level 0 and level 2. In the following paragraphs, we describe them in more detail.

0 Extremal Dynamics: At time T the agent with the smallest utility is removed from the system and replaced with a new one (market entry). The new agent is randomly connected to the already existing agents and a small initial value of knowledge is assigned to it. This process is a least fit replacement (extinction of the weakest) and the new agent introduces a kind of novelty in the system (innovation).

2 Utility Driven Dynamics: The main difference between local link formation (utility driven dynamics) compared to global link formation (extremal dynamics) is that agents are now individually taking decisions upon their interactions and they do that on the basis of a utility function (their values of knowledge at time T). Agents are bounded rational since they explore their possible interaction partners in a trial and error process. At every period, that is after time T, an agent is selected at random to create and delete links (asynchronous update). We distinguish two possible link formation mechanisms which we study separately, namely unilateral and bilateral link formation. In the former, unilateral link formation (i), the agent optimally deletes an old link and randomly creates a new link. Optimal means that either for creation or deletion of links the action is taken only if it increases the value of knowledge of the agent at time T in the range of all possible actions. In the latter, bilateral formation (ii), the selected agent optimally deletes a bilateral connection that she currently has or she randomly creates a new bilateral connection. Here optimal (in the range of all possible actions) means, that links are deleted if the initiator of the deletion, i.e., the selected agent, can increase its value of knowledge at time T with the deletion of the link, while for the bilateral creation both agents involved have to strictly benefit from the creation of the mutual connection.[34] In the following two sections, we give a description of mechanisms (i) and (ii).

To compare the two levels, for utility driven dynamics the evolution of the network follows from local, utility driven, actions, as opposed to extremal dynamics, where the evolution follows from a global stochastic process (least fit selection plus random link formation). To be more specific about the latter, the rules for the network evolution, i.e., the creation and deletion of links under extremal dynamics, are the following:

Step 1 After a given time T the *least fit agent*, i.e., the one with the smallest $u_i = y_i(T)$, is determined. This agent is removed from the network along with all its incoming and outgoing links.

Step 2 A new agent is added to the network with some small initial value of knowledge y_0. The new agent will take the place of the old one (it gets the same label), and randomly links itself to the other nodes in the network with the same probability p. Each of the other nodes can in turn link itself to the newcomer node with a probability p.

These rules for the network evolution are intended to capture two key features: *natural selection*, in this case, the extinction of the weakest; and the *introduction of novelty*. Both of these can be seen as lying at the heart of natural evolution. The particular form of selection used in this model has been inspired by what *Bak and Sneppen* have called "extremal dynamics" (Bak and Sneppen, 1993).

8.3.9.3 Rules for Link Creation and Deletion Using Utility Driven Dynamics

In this section, we introduce the process of the formation and deletion of links by agents that maximize a local utility function (depending on the agent and its neighbors). After time T, long enough such that the system reaches a quasi-equilibrium in the values of knowledge, an agent is randomly chosen to create or delete a link, either unidirectional or bidirectional.

Unilateral Link Formation

If agents unilaterally delete or create links, it is possible that the interactions they form create a feedback loop, i.e., a closed cycle of knowledge sharing agents, that involves more than two agents. This introduces the concept of indirect reciprocity (see Sect. 8.3.3). Unilateral formation of links (we then have a directed network) is necessary for indirect reciprocity to emerge, since if all interactions were bilateral they would be direct reciprocal by definition. We now describe the procedure of unilateral link creation and deletion.

[34] This behavior is individually optimal and thus may also be called rational.

1. *Random Unilateral Creation*
   An agent creates a link to another one to which it is not already connected at random and evaluates the creation of the link by comparing the change in their values of knowledge before and after the creation. Only if the change is positive, the link is maintained, otherwise the agent does not create the link. In this way agents explore possible partners for sharing their knowledge in a trial and error procedure (Fig. 8.8).

   Step 1 An agent i is selected at random.
   Step 2 Another agent j is selected at random which is not already an out-neighbor of i.
   Step 3 Agent i creates an outgoing link to agent j.
   Step 4 The new utility (for the old network plus the new link e_{ij}) of agent i is computed and compared with the utility before the creation.
   Step 5 Only if agent i's utility strictly increases compared to her old utility, then the link is created.

2. *Optimal Unilateral Deletion*
   An agent deletes one outgoing link if this increases her utility (Fig. 8.9).

   Step 1 Agent i is selected at random s.t. it has at least one outgoing link.
   Step 2 Agent i deletes separately each of its outgoing links to its neighbors $v_j \in N_i^+$ and records the change in her utility, Δu_i. Before the next link is deleted, the previous one is recreated.
   Step 3 Agent i computes the maximum change Δu_i and if it is positive, deletes the referring link. This means that only one link is finally deleted. The deletion only takes place if the current agent strictly increases her utility.

To characterize the equilibrium networks under this link formation and deletion mechanism, we introduce the following characterization of stability.[35]

Definition 24 A network is **unilaterally stable** if and only if (i) no agent can create a link to (strictly) increase her utility and (ii) no agent can remove a link to (strictly) increase her utility.

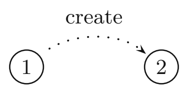

Fig. 8.8 Random unilateral creation

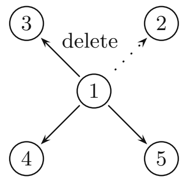

Fig. 8.9 Optimal unilateral deletion

[35] Compare this to the definition of bilateral stability (8.20)

Bilateral Link Formation

If agents form links bilaterally then all interactions are direct reciprocal by definition. We describe the process of bilateral link creation and deletion in the following paragraphs:

1. *Random Bilateral Creation*
   In this link creation process, a pair of agents is selected at random and given the possibility to form a bilateral connection (Fig. 8.10).

   Step 1 Two agents are uniformly selected at random such that they are not connected already.
   Step 2 Both agents create an outgoing link to each other and therewith create a 2-cycle.
   Step 3 The new utilities (for the old network plus the new 2-cycle) of both agents are computed and compared with the utilities before the creation.
   Step 4 Only if both agents strictly benefit in terms of their utilities compared to their old utilities, then the bilateral connection is created.

2. *Optimal Bilateral Deletion*
   An agent deletes one of its outgoing links to another agent from which the agent also has an incoming link if this deletion increases her utility (Fig. 8.11).

   Step 1 Agent i is selected at random such that it has at least one mutual link to another agent.
   Step 2 From all bilaterally connected neighbors agent i deletes separately each of its outgoing links to its neighbors (and so does each neighbor j to agent i). For each, the change in the utility, Δu_i is recorded. Before new links are deleted, the old ones are recreated.
   Step 3 Agent i computes the maximum change Δu_i and, if it is positive, the referring bilateral connection is deleted. The deletion only takes place if agent i strictly increases her utility.

In order to characterize the equilibrium outcomes of our simulations, we will introduce a characterization of network stability. This definition has been introduced already in Sect. 8.2.2 and we repeat it here for expository reasons.

Definition 25 A network G is *pairwise stable* if (i) removing any link does not increase the utility of any agent and (ii) adding a link between any two agents, either doesn't increase the utility of any of the two agents, or if it does increase one of the two agents' utility then it decreases the other agent's utility.

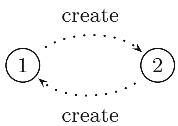

Fig. 8.10 Random bilateral creation

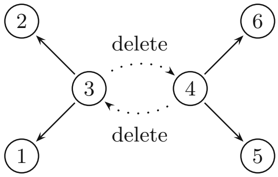

Fig. 8.11 Optimal bilateral deletion

8.3.9.4 The Role of the Time Horizon for Unilateral Link Formation

So far, we have assumed that the time horizon T (after which agents evaluate their decisions to create or delete links) is long enough such that the values of knowledge reach a stationary state and the utilities of the agents are given by the fixed points of the values of knowledge. In this section, we discuss the effect of a time horizon that is smaller than the time to convergence to the stationary state of the values of knowledge. For related works that incorporate a finite time horizon in the evaluation of the actions of agents see, e.g., Huberman and Glance (1994) or Lane and Maxfield (1997).

If we consider utility driven dynamics, we will show that permanent networks with positive values of knowledge emerge if agents wait long enough (with respect to the time the values of knowledge need in order to reach a stationary state) in evaluating their decisions. This is a necessary condition. Otherwise networks are not able to emerge or, if a network with positive knowledge values is existing already, it gets destroyed over time (network breakdown). This effect is important in the case of null as well as increasing costs.

To illustrate this point, we consider a 5-cycle of agents and the deletion of one link in this cycle which creates a linear chain of five nodes, Fig. 8.12. The evolution of value of knowledge for null costs and for costs $c = 0.5$ can be seen in Fig. 8.13.

More formally we can give the following proposition:

Proposition 26 *Consider the dynamical system* (8.21). *For a directed path P_k of length k the value of knowledge of node k is larger than ϵ for $t \leq \tau(\epsilon)$, i.e., $x_k(t \leq \tau(\epsilon)) \geq \epsilon$ while* $\lim_{t\to\infty} x_k(t) = 0$.

Proof Consider a directed path P_k of length k (Fig. 8.14).

For node 1 (the source has no incoming links) in (8.21) we get

$$\dot{x}_1(t) = -dx_1 - cx_1^2. \tag{8.72}$$

By introducing the variable $z = \frac{1}{x_1}$ and solving for z, one can find the solution for x_1

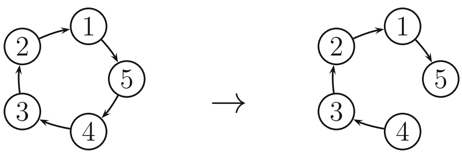

Fig. 8.12 A 5-cycle and a linear chain of five nodes (obtained from the cycle by removing one link)

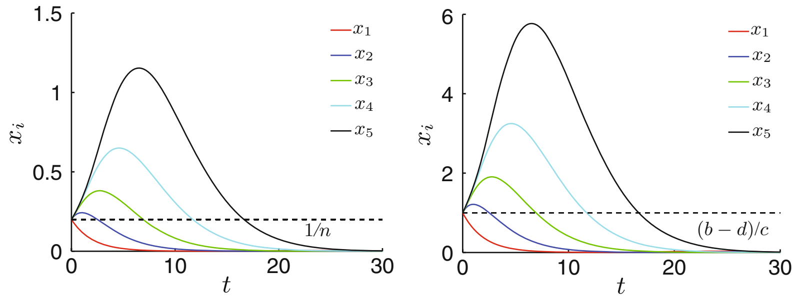

Fig. 8.13 Numerical integration of the value of knowledge for $d = 0.5$, $b = 1.0$, null cost $c = 0.0$ (*left*) and cost $c = 0.5$ (*right*): evolution of knowledge values for a linear chain of five nodes (obtained from the C_5 by removing a link). The agent that removes the link (black upper curve) initially experiences an increase in the value of knowledge. After an initial increase she experiences a decline and at a certain time her value of knowledge reaches her initial value ($1/n$ in the case of null cost and $(b-d)/c$ in the case of increasing cost) and then it further decreases. After a time long enough her value of knowledge vanishes completely

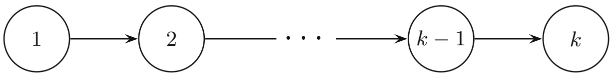

Fig. 8.14 A directed path P_k of length k

$$x_1(t) = \frac{de^{da}}{e^{dt} - ce^{da}} \tag{8.73}$$

with a constant $a = \frac{1}{d} \ln \frac{x_1(0)}{d+c}$ and the limit $\lim_{t\to\infty} x_1(t) = 0$. Accordingly, for the k-th node we have that

$$\dot{x}_k = -dx_k + bx_{k-1} - cx_k^2. \tag{8.74}$$

Since $x_k \geq 0$, from Proposition (18), the following inequality holds

$$\dot{x}_k \geq -dx_k - cx_k^2 \tag{8.75}$$

and

$$x_k(t) \geq \frac{de^{da'}}{e^{dt} - ce^{da'}} \tag{8.76}$$

with a proper constant $a' = \frac{1}{d} \ln \frac{x_k(0)}{d+c}$. Equating with $\epsilon(t)$ at $t = \tau$ we get

$$\epsilon = \frac{d}{e^{dt-da'} - c} \tag{8.77}$$

which yields

$$\tau(\epsilon) = \frac{\ln\left(\frac{d}{\epsilon} + c\right) + a'd}{d}. \tag{8.78}$$

Thus, we have found an $\epsilon(\tau)$ such that for $t \leq \tau(\epsilon)$ $x_k(t) \geq \epsilon$. The limit $\lim_{t\to\infty} x_k(t) = 0$ follows directly from the fact that the directed path P_k is a directed acyclic graph and we can apply Proposition (15). □

With Proposition (26) one can readily infer the following. If an agent in a cycle C_k of length k removes a link unilaterally then a path P_k is created. If the time horizon after which the agent evaluates this link removal is smaller then $\tau(\epsilon)$) the agent's value of knowledge satisfies $x_k(t \leq \tau(\epsilon)) \geq \epsilon$ (this gives the utility of the agent, see (23)). From Proposition (20) we know that the value of knowledge of the agent in the cycle is given by $x_k(0) = \frac{b-d}{c}$. Choosing $\tau(\epsilon)$ such that $\epsilon > \frac{b-d}{c}$ gives $x_k(t \leq \tau(\epsilon)) \geq x_k(0)$ and the agent experiences an increase in her utility by removing the link. The agent removes the link in order to increase her utility. This destroys the cycle. The time horizon of the agent in this case is too short in order to anticipate the vanishing long-run values of knowledge of all the agents in the resulting path, $\lim_{t\to\infty} x_(t) = 0$.

From this observation we conclude that if the time horizon is too short, then all cycles would get destroyed and no network would ever be able to emerge nor sustain, since only cyclic networks can be permanent. Agents who remove their links because, in the short run, their utility increases therewith, can be considered as free-riders. The value of knowledge is maintained by their predecessors in the cycle while they refuse themselves to contribute to knowledge sharing and production in the network since they do not have any outgoing links. In the short run they benefit from the knowledge shared and produced in the network without contributing to it. However, as we have seen from the discussion above, in the long run this causes the total value of knowledge of the network to vanish. Therefore, we can say that the free-riding behavior of agents leads to the breakdown of the economy.

8.3.9.5 Simple Equilibrium Networks for Unilateral Link Formation

In this section, we identify the most simple equilibrium networks for unilateral link formation. There exists a multitude of other equilibrium networks which usually cannot be computed analytically and which depend on the parameter values for decay, benefit, and cost.

The most simple equilibrium network is the empty network.

Proposition 27 *The empty graph is unilaterally stable.*

Proof In an empty graph all nodes have vanishing values of knowledge. Creating a link does not create a cycle (which would be the case however if links were formed bilaterally) and thus the empty graph plus a link is a directed acyclic graph with vanishing values of knowledge, see Proposition (15). The creation of a link does not increase the utility of an agent. Thus, the agents do not form any links. □

Moreover, if all agents form disconnected cycles then we have an equilibrium network.

Proposition 28 *The set of disconnected cycles* $\{C^1, ..., C^k\}$*, and possibly isolated nodes is unilaterally stable.*

Proof We give a proof for 2-cycles. The proof can easily be extended to cycles of any length. Consider the two cycles C_2^1 and C_2^2 in Fig. 8.15.

From Proposition (20) we know that the fixed points are given by $x_i = \frac{b-d}{c}$, $i = 1, ..., 4$. In order to show that we have a unilaterally stable equilibrium, we (i) first show that no link is created and in the following (ii) that no link is deleted.

(i) If a link is created (w.l.o.g.) from node 2 to 4 we get from the dynamics on the value of knowledge in the case of increasing interaction costs given by (8.21)

$$\begin{aligned} \dot{x}_1 &= -dx_1 + bx_2 - cx_1^2 \overset{!}{=} 0 \\ \dot{x}_2 &= -dx_1 + bx_1 - cx_2^2 \overset{!}{=} 0 \end{aligned}. \tag{8.79}$$

From the first order conditions for the fixed points we get for node 1

$$x_1 = \frac{bx_2 - c}{d}. \tag{8.80}$$

And inserting this into the fixed point of node 2 gives

$$x_2 = \frac{b^2 - d^2 + \sqrt{b^4 - 8bc^3d - 2b^2d^2 + d^4}}{4cd} \tag{8.81}$$

If the last inequality is fulfilled, then the creation of the link would decrease the utility of agent 2. The inequality holds if $c^3 \geq \frac{(d-b)^3}{b}$ which is certainly true for $b > d$ and $c > 0$. Thus, no link is created between the cycles.

(ii) If a link is deleted in a C_2 then we get vanishing steady-state values of knowledge. Since $\frac{b-d}{c} \geq 0$ this would reduce the utility of the agent. Therefore, the link is not removed.

If there are $k \leq \lfloor \frac{n}{2} \rfloor$ 2-cycles in G then the above argument holds for any pair of cycles. Similarly, no isolated node can create a link in order to increase her utility nor can a node in a cycle create a link to an isolated node. Neither link creation

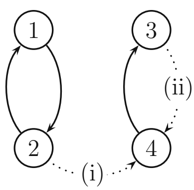

Fig. 8.15 Two cycles C_2^1 and C_2^2 and the cases of link creation (i) and deletion (ii)

nor removal increases the utility of the initiating agent and so the set of 2-cycles is unilaterally stable. □

We further conjecture that a set of disconnected autocatalytic sets, where an autocatalytic set is defined as a set of nodes each having an incoming link from a node of that set (Jain and Krishna, 2001), stays disconnected under unilateral link formation. Thus, the size (in terms of nodes) is stable.

With (28) we know that a cycle is unilaterally stable. In Sect. 8.3.9.4, however, we have shown that this result is critically depending on the time horizon T after which the action of an agent is evaluated (and it is true for cycles of any length only if $T \to \infty$).

For parameter values $d = 0.5$, $b = 0.5$, and $c = 0.1$ also, the complete graph with three nodes K_3 and the path P_3 is unilaterally stable. We observe this in simulations in Fig. 8.22. However, by computing the fixed points numerically for $d = 0.5$, $b = 0.5$, and $c = 0.1$ in Appendix (A) one can see that K_3 is no longer unilaterally stable (because removing a link increases the utility of an agent).

In the next section, we investigate if the dynamic processes of link formation and deletion lead to the simple equilibrium structures suggested above (and indeed we show that they are not obtained).

8.3.9.6 Simulation Studies Using Different Growth Functions

In the remainder of this chapter, we study simulations with different growth functions (for the value of knowledge) and different link formation mechanisms. We assume that the time horizon T is long enough such that the values of knowledge reach their stationary state. The dynamics of the value of knowledge is given by (8.19) with null costs or by (8.21) with increasing costs. The different link formation mechanisms are described in Sect. 8.3.9.2. We compare the equilibrium networks obtained from different costs and link formation rules in terms of their structure and performance. Finally, we study the effect of different positive network externalities on the equilibrium networks.

Table 8.3 gives an overview of the simulations that we study in the following. We set $d = 0.5$, $b = 0.5$, and $c = 0.1$. The complete set of parameter values used throughout this section can be found in Table 8.5 in the Appendix (C).

8.3.9.7 Null Interaction Costs

In the following, we briefly discuss the evolution of the network with least fit link formation and null link costs. This model has been studied in detail by Seufert and Schweitzer (2007); Jain and Krishna (2001). Later Saurabh and Cowan (2004) have applied it to an innovation model where new ideas are created and destroyed in a network of ideas.

In this model agents do not have to pay costs for maintaining interactions. Accordingly, the dynamics on the values of knowledge is given by (8.19)

Table 8.3 Overview of the simulation studies in the following sections with different knowledge growth functions and different link formation mechanisms

Knowledge Dynamics	Network Dynamics	Section
Null costs, $c_{ij} = 0$: $\frac{dx_i}{dt} = -dx_i + b\sum_{j=1}^{n} a_{ji}x_j$	Least fit replacement	(8.3.9.7)
Quadratic cost, $c_{ij} \propto x_i^2$: $\frac{dx_i}{dt} = -dx_i + b\sum_{j=1}^{n} a_{ji}x_j - c\sum_{j=1}^{n} a_{ij}x_i^2$	Least fit replacement unilateral link formation bilateral link formation	(8.3.9.8)
Quadratic cost, $c_{ij} \propto x_i^2$, and externality, w_{ji}: $\frac{dx_i}{dt} = -dx_i + \sum_{j=1}^{n}(ba_{ji} + b_e w_{ji})x_j - c\sum_{j=1}^{n} a_{ij}x_i^2$	Unilateral link formation	(8.3.9.9)

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -dx_i + b\sum_{j=1}^{n} a_{ji}x_j$$

and the dynamics in the shares of the values of knowledge $y_i = x_i/\sum_{j=1}^{n} x_j$ is given by (8.48).

$$\dot{y}_i = \sum_{j}^{n} a_{ji}y_j - y_i\sum_{k,j}^{n} a_{jk}y_j.$$

The utility is given by $u_i = y_i(T)$. We have described in Sect. 8.3.8 that the fixed point (stationary solution) of the relative values of knowledge in (8.48) exists and is given by the eigenvector to the largest real eigenvalue of the adjacency matrix. We assume that the time horizon T (after which links get created or deleted) is large enough such that the system has reached this stationary state before links are changed.

Extremal Dynamics: Least Fit Replacement

After time T the worst performing agent (in terms of her share of value of knowledge $y_i(T)$) is replaced with a new one. We have described this global link formation mechanism in Sect. 8.3.9.2. Jain and Krishna (1998a, 2001), Seufert and Schweitzer (2007) have extensively studied the behavior of the dynamics on $\mathbf{y}$ and the network G represented by $\mathbf{A}(G)$. They showed that strongly connected sets of nodes with free-riders (that are receiving knowledge from the strong component but are not contributing knowledge back to the strong component) attached[36] appear and get destroyed in the process of repeatedly removing the worst performing node (with minimum y_i) and replacing it with a new one.

[36] Jain and Krishna (2001) denote this set of nodes the *autocatalytic set (ACS)*: it is a subgraph of nodes in which every node has at least one incoming link from that subgraph.

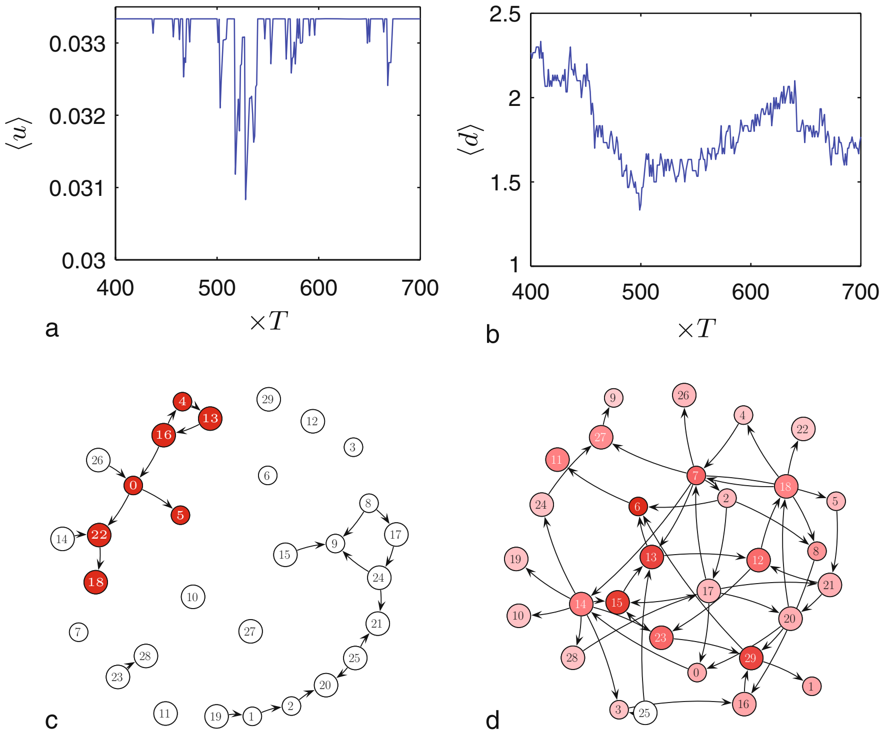

Fig. 8.16 Least fit replacement: (**a**) Average utility. (**b**) Average degree. (**c**) Initial random graph. (**d**) Graph after 5000 iterations

In computer simulations we can reproduce the results of Jain and Krishna (2001), Seufert and Schweitzer (2007). We observe crashes and recoveries in the average utility and degrees of the agents over time as can be seen in Fig. 8.16. Thus, no stable equilibrium network can be realized with this type of network dynamics.

In the model of Jain and Krishna (2001) links are costless. In the next Sect. 8.3.9.8 we assume that links have a cost attached, that is an increasing function of the value of knowledge that is being transferred (Sect. 8.3.9.8).

Moreover, the least fit network dynamics treats agents as completely passive units that are exposed to an external selection mechanism. In a more realistic approach one should take into account that agents are deliberately deciding upon with whom to engage in an R&D collaboration or to share their knowledge with. These decisions are taken on the basis of increasing a utility function, that is their value of knowledge.[37] We introduce local link formation rules in Sect. 8.3.9.3. Moreover, as a further extension we study the effect of positive network externalities in Sect. 8.3.9.9.

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[37] A model in which the eigenvector associated with the largest real eigenvalue is used as a utility function is studied in Ballester et al. (2006).

8.3.9.8 Increasing Interaction Costs

In this section we study the effect of increasing costs for maintaining interactions with other agents on the resulting equilibrium networks. The evolution of the value of knowledge is given by (8.53) and the utility of the agents by (8.18). The cost of a link depends quadratically on the value of knowledge of the agent that initiates the interaction. We study three different link formation mechanisms. The first is a least fit replacement. We will compare the results of the simulation with the preceding section where links were costless. In the following two sections link formation mechanisms are studied in which agents decide locally upon to create or delete links either unilaterally or bilaterally based on their utility (8.18). We assume that the time horizon is long enough such that the utility of the agents is given by the fixed points of the value of knowledge. We will show that least fit replacement of agents leads to a total network breakdown eventually from which the system cannot recover. Moreover, we show that bilateral link formation leads to a complete graph while with unilateral link formation this is not the case. For unilateral link formation only a small number of agents have non-vanishing knowledge values in the resulting equilibrium network and these cluster together in bilateral connections. Depending on the link formation mechanism and the parameter values (for decay, benefit, and cost) the equilibrium networks can vary considerably.

The evolution of the value of knowledge of agent i (8.53)

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -dx_i + b\sum_{i=1}^{n} a_{ji}x_j - c\sum_{i=1}^{n} a_{ij}x_i^2$$

and her utility is given by $u_i = x_i(T)$.

Extremal Dynamics: Least Fit Replacement

Similarly to the preceding section, links are formed and removed by a least-fit selection mechanism (introduced in Sect. 8.3.9.2). The agent with the smallest utility (8.18) is replaced with a new agent. But in this section costs for maintaining links are an increasing function of the knowledge value of the transmitting agent.

In this setting, it is possible that the system breaks down completely. A simulation run exhibiting such a crash can be seen in Fig. 8.17. If the network is sparse enough the link removal mechanism can destroy the cycles in the network and thus creates a directed acyclic graph. As soon as the network evolution hits a directed acyclic graph, all value of knowledge vanish (and accordingly the utilities of the agents) and the network entirely breaks down.

We do not experience a breakdown of the network in the case of null costs in the last section since there we were considering relative values of knowledge only. The normalization of the relative values, $\sum_{i=1}^{n} y_i = 1$ prevents all the shares to become 0 at the same time, $y_i = 0\ \forall i$. Thus, we do not get a total breakdown of the network in which all values of knowledge vanish. Instead, there the system can always recover from a crash of the network.

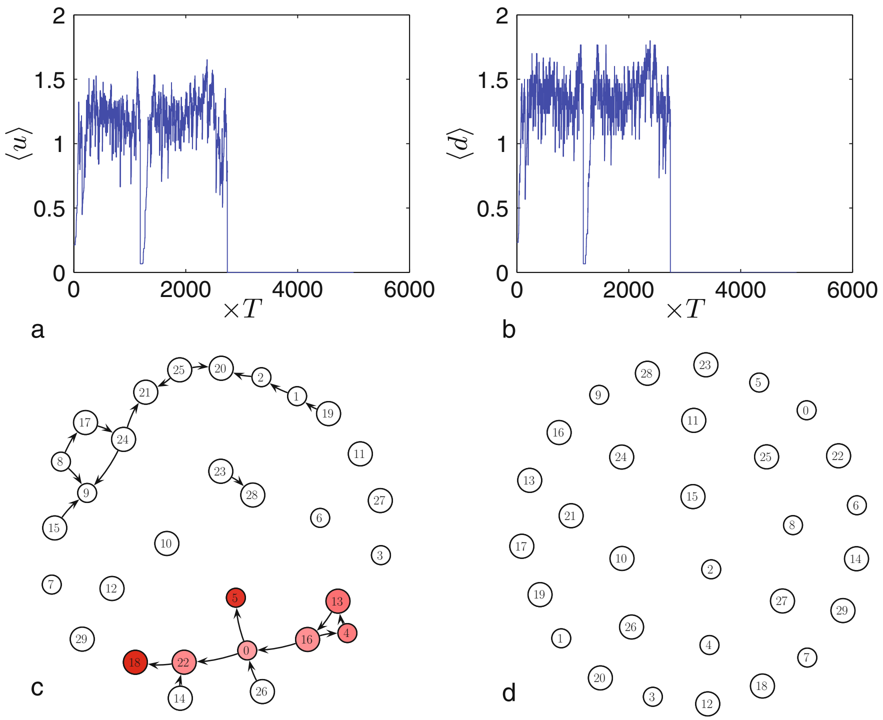


Fig. 8.17 Extremal dynamics: (**a**) Average utility. (**b**) Average degree. (**c**) Initial random graph. (**d**) Graph after 5000 iterations (in the equilibrium). The network experiences a total breakdown eventually

Utility Driven Dynamics: Bilateral Link Formation

In this section, agents are creating or deleting links bilaterally. All interactions are therefore direct reciprocal. In simulations we observe the following effect. Bilateral creation and deletion results in a complete subgraph (the average degree is $1/n \sum d_i = 1/20 \times 8 \times 7 = 2.8$, see Fig. 8.20) of the agents that were part of a permanent set in the initial graph[38] (the stability criterion which defines an equilibrium network is given in (25)).

Utility Driven Dynamics: Unilateral Link Formation

The mechanisms of unilateral creation and deletion of links has been introduced in Sect. 8.3.9.3. In our simulations we observe the following effect. When we allow for unilateral link formation, large cycles get reduced to a small set of 2-cycles. In the equilibrium network (the stability criterion which defines an equilibrium network is

[38] The creation of the initial random graph with a given link creation probability has been chosen rather small such that only a few nodes are permanent.

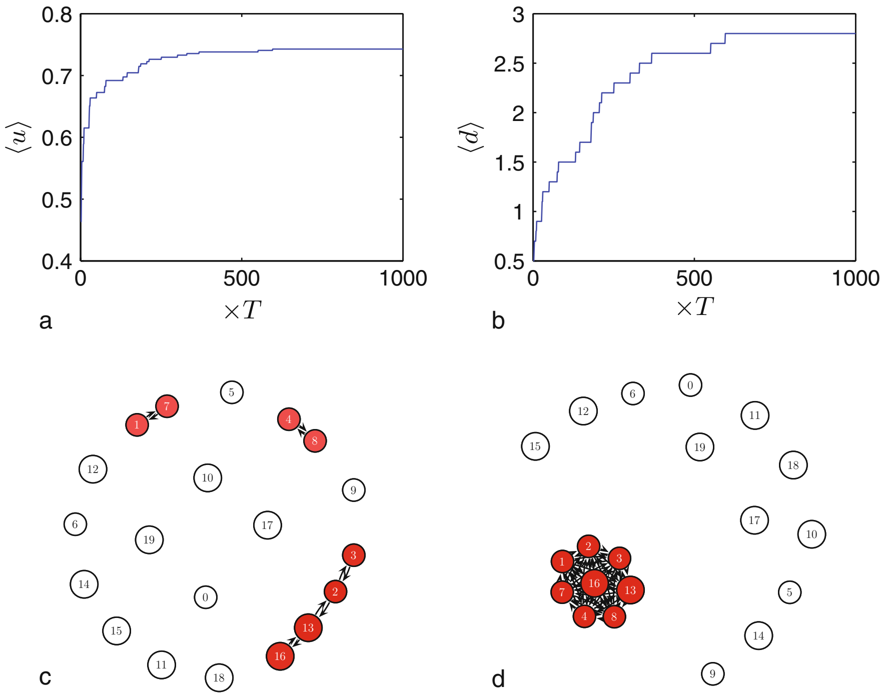


Fig. 8.18 Bilateral link formation: (**a**) Average value of knowledge. (**b**) Average degree. (**c**) Initial random graph (for reasons of visualization we have chosen a rather sparse random graph). (**d**) Graph after 1000 iterations (in the equilibrium)

given in (8.19)) most of the agents are isolated nodes and thus have vanishing values of knowledge. Only a few of them are organized in 2-cycles and small subgraphs consisting of multiple 2-cycles. As we will show, the reason for this is that as soon as there exists a shortcut (a smaller cycle) in a larger cycle agents try to free-ride and, after the other agents have realized that and sopped sharing their knowledge with them, they get isolated and experience vanishing values of knowledge. One can interpret this result as follows: Even though agents could in principal form indirect reciprocal interactions the resulting equilibrium network consists only of direct reciprocal interactions (2-cycles and clusters of 2-cycles).

We can give an example of the process of the reduction of cycles in a graph G with three nodes for parameter values $d = 0.5$, $b = 1$, and $c \in (0, 1)$. By numerically comparing utilities (the fixed points of the value of knowledge) before and after a link is created or deleted, we show that there exists a sequence of link deletions and creations which transform a 3-cycle into a 2-cycle while every link change is associated with an increase in the utility (the fixed point in the value of knowledge) of the initiating agent (Jackson (2003) calls this sequence of graphs an "improving path").

In Fig. 8.19 (left) agent 3 creates a link to agent 1 because in this range of parameters this increases her value of knowledge. This can be seen in Fig. 8.19

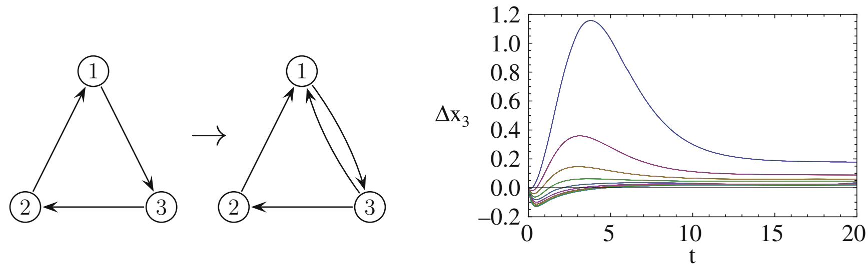


Fig. 8.19 Agent 3 forms a link to agent 1, e_{31}, and thus a 2-cycle is created inside a 3-cycle. The situation is illustrated on the *left hand side*. On the *right*, the evolution of the values of knowledge for different values of cost is shown

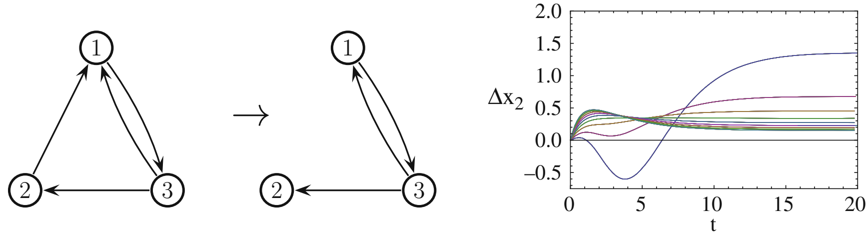


Fig. 8.20 Deletion of the link e_{21}. Agent 2 is not sharing any knowledge with others but only receiving knowledge from agent 3. Thus, agent 2 is free-riding. The situation is illustrated on the *left hand side*. On the *right*, the evolution of the values of knowledge for different values of cost are shown

(right), where the increase Δx_3 different costs $c \in (0, 1)$ are plotted and $\Delta u_3 = \lim_{t\to\infty} \Delta x_3 > 0$.

In Fig. 8.20 (left) agent 2 removes her link to agent 1 and thus she stops contributing knowledge but instead is only receiving knowledge from agent 3. We say that agent 2 is free-riding. This increases her utility, since $\Delta u_2 = \lim_{t\to\infty} \Delta x_2 > 0$, as can be seen in Fig. 8.20 (right) for different costs c.

Finally, in Fig. 8.21 (left) agent 3 removes her link to agent 2 because she is better off, as illustrated in Fig. 8.21 (right), when she stops contributing knowledge

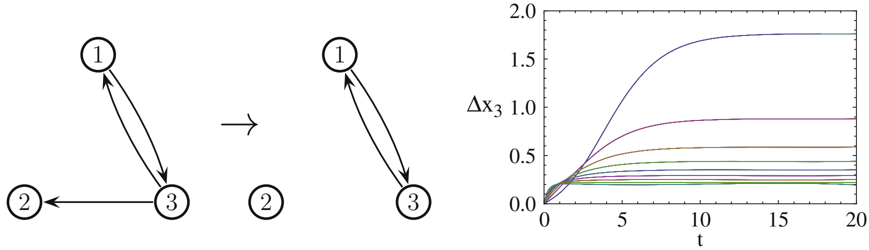


Fig. 8.21 Deletion of the link e_{32} by agent 3. Agent 3 realizes that she is better off by not sharing her knowledge with agent 2. Agent 2, who was free-riding before now gets isolated and experiences a vanishing value of knowledge in the long run. The situation is illustrated in the figure to the left. On the right, the evolution of the values of knowledge for different values of cost are shown

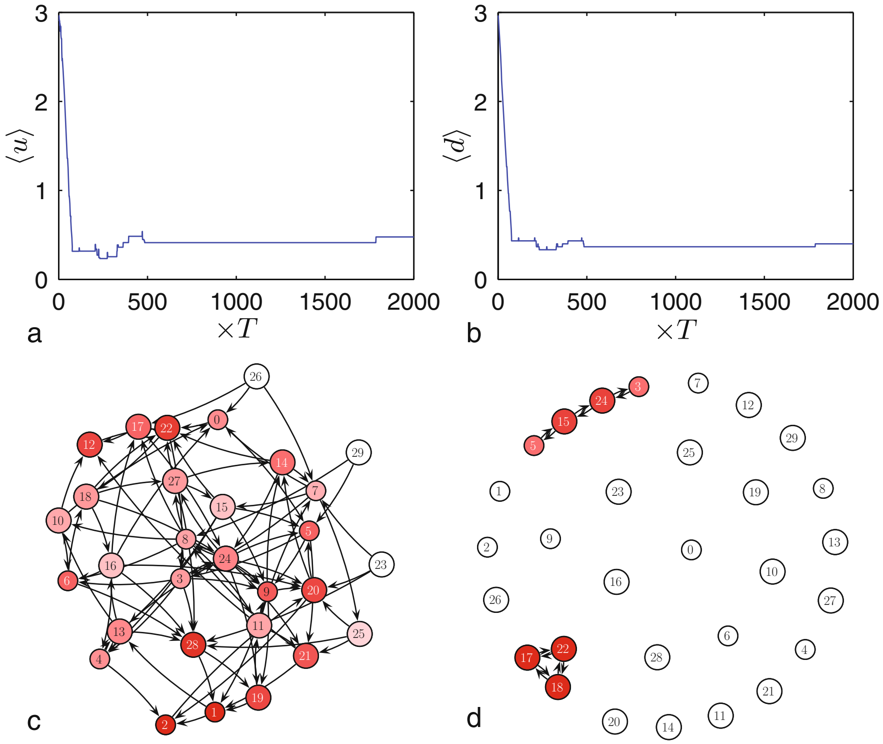


Fig. 8.22 Unilateral link formation: (**a**) Average utility. (**b**) Average degree. (**c**) Initial random graph. (**d**) Graph after 2000 iterations (in the equilibrium). For the parameter values $d = 0.5$, $b = 0.5$, $c = 0.1$, used in this simulation, the complete graph K_3 is an equilibrium. Note from Appendix (A) one can see that for parameter values $d = 0.5$, $b = 1$, $c = 0.5$ this is no longer the case and K_3 would be reduced to a 2-cycle C_2

to an agent that is nothing contributing in return. This is actually true for agent 2. Agent 2 therefore gets isolated and experiences a vanishing value of knowledge in the long run, $\lim_{t\to\infty} x_2 = 0$. Her utility is null.

We end up in a setting where out of a cooperation of many (the sharing of knowledge) only a small set of cooperators remains and all the remaining agents vanish, i.e., have vanishing values of knowledge and utility. We can see this in a simulation starting from an initial random graph with 30 agents and the resulting equilibrium network in Fig. 8.22 (bottom right).

Since the performance of the system in terms of the total value of knowledge is very low, we investigate in the next section the conditions under which the performance can be increased (with more agents being permanent in the equilibrium). We find that the existence of a positive network externality (explained in the next section) can enhance the performance of the system.

8.3.9.9 Introducing Positive Network Externalities

In this section, we study the growth of the value of knowledge which includes an additional benefit term contributing to an increase in the value of knowledge. This additional benefit depends on the network structure itself. In the economic literature (Mas-Colell et al., 1995; Tirole, 1988), "positive network externalities arise when a good is more valuable to a user the more users adopt the same good or compatible goods." In our model we define a network externality simply as a function of the network structure that affects the utility of an agent. Including the externality in the benefit can yield more complex structures with non-vanishing knowledge values as equilibrium networks. More precisely, we introduce weights for the connections between the agents that depend on a measure of network externalities that we will introduce in the following sections. This means that, if we have strong network externalities between agent i and agent j, then the weight w_{ij} will represent this effect and attain a high value. Taking into account the existence of such network externalities, the growth of the value of knowledge of agent i is given by the following equation:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -dx_i + b\sum_{i=1}^{n} a_{ji}x_j + \underbrace{b_e\sum_{i=1}^{n} w_{ji}x_j}_{\text{positive network externality}} - c\sum_{i=1}^{n} a_{ij}x_i^2 \tag{8.82}$$

and the utility is again given by $u_i = \lim_{t\to\infty} x_i(t)$. Link changes are based on the increase in utility. The network benefit incorporates the fact that the value of knowledge can change with the number of users of that knowledge, (8.82). But the number of users can either enhance or diminish the value of knowledge that is being transferred between agents, depending on the type of knowledge under investigation. On one hand, the value can decrease with the number of agents that pass on that knowledge. Knowledge is attenuated with the distance from the creator to the receiver. We study this type of knowledge with a link weight defined in (8.82) and denoted by w^{c}_{ji}. In the next section, we study the opposite effect: the value of knowledge increases with the number of users. This holds for example for general purpose technologies that get more valuable the more they are applied and used in different contexts (and users). The link weights used for this type of knowledge in (8.82) are denoted by w^{ccn}_{ji}, w^{cce}_{ji}, where the first measures the number of agents using that knowledge and the second the number of interactions.

We introduce different link weights, denoted by w^{c}_{ji}, w^{ccn}_{ji}, w^{cce}_{ji}. Moreover, agents are creating and deleting links unilaterally (utility driven dynamics). We then study the effect of different weights on the equilibrium networks obtained.

Shortest-Path Centrality

The growth function of the value of agent i is given by

$$\frac{dx_i}{dt} = -dx_i + \sum_{j=1}^{n}(ba_{ji} + b_e w^{\mathrm{c}}_{ji})x_j - c\sum_{j=1}^{n} a_{ij}x_i^2. \tag{8.83}$$

The utility is given by $u_i = \lim_{t\to\infty} x_i(t)$ (large T). Link changes are accepted on the basis of an increase in utility. The shortest-path centrality measure computes the sum of the inverse lengths of all the shortest paths containing the link for which the centrality is computed. If two agents are not connected then the length of the path is assumed to be infinity and thus its weight is zero. Instead, if two agents are directly connected via a link, then the weight is one. The weight values links more that bring agents closer to each other. This is a similar approach to the Connections Model introduced in Sect. 8.2.2 with a utility given by (8.4). The centrality link weight, w^{c}_{ij}, is then computed as follows:

$$w^{\mathrm{c}}_{ij} = \sum_{v\in V} \frac{1}{(d_{jv}+1)},\ w^{\mathrm{c}}_{ij} \in [0, 1], \tag{8.84}$$

d_{jv} is the shortest path between node j and node v. If there exists no path between two nodes, then the distance between them is infinity.[39]

In simulations, Fig. 8.23, we observe that in the equilibrium network only a few agents have non-vanishing utilities (the asymptotic knowledge values) and most of them are isolated nodes with zero utilities. This result does not differ too much from the studies in Sect. 8.3.9.8 where no externality is considered. Apparently, if more agents should have non-vanishing utilities induced by an additional benefit depending on the network structure, this cannot be realized with the centrality link weight.

Circuit-Centrality

The *circuit-centrality* measure puts a weight on the links that depends on the number of distinct nodes that are contained in all the circuits going through the link under consideration. The motivation is that, if many agents are involved in the transfer of knowledge and this knowledge then comes back to the agent (thus creating a feedback on the technology issued by the agent), it gets an added value (e.g., for general purpose technologies (GPT) (Bresnahan and Trajtenberg, 1995; Karshenas and Stoneman, 1995; Cohen, 1995)). The more agents use a technology the more it is improved and so the more agents are involved in such a feedback loop the higher is the value of the technology. We can either count the number of different agents involved in this feedback loop or the number of interactions (links). Either possibility is explored in the next sections. This is an alternative way to study the

[39] We use a standard *depth-first-search* algorithm to compute the shortest paths. More details on this algorithm and further discussion is given in Ahuja et al. (1993), Cormen et al. (2001), Steger (2001), Steger and Schickinger (2001).

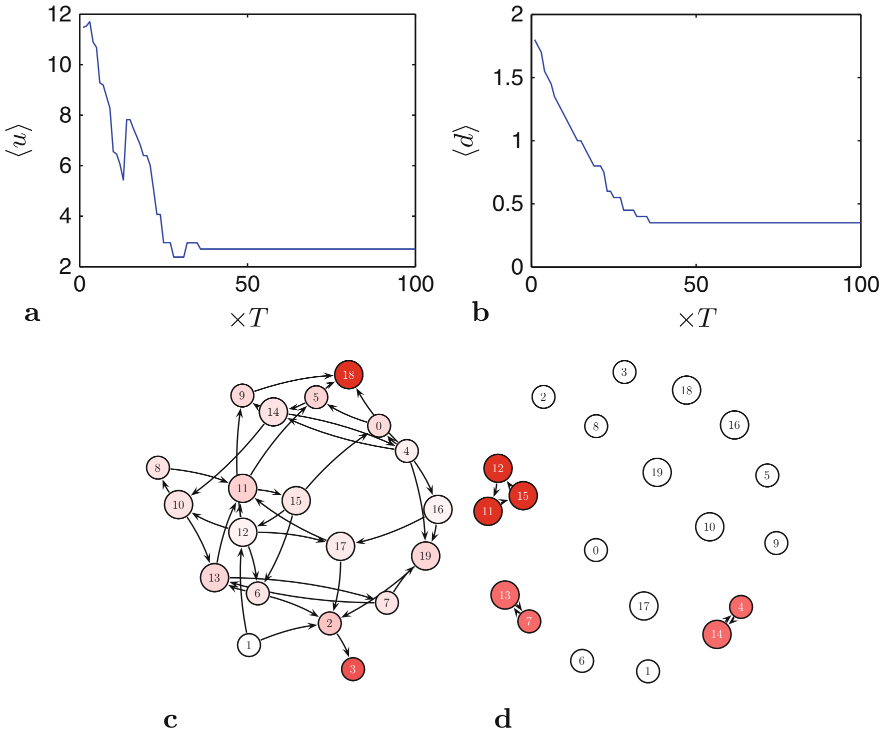


Fig. 8.23 Shortest-path Centrality: (**a**) Average utility. (**b**) Average degree. (**c**) Initial random graph. (**d**) Graph after 500 iterations (in the equilibrium)

emergence of indirect reciprocity where others Nowak and Sigmund (2005) have studied it by introducing a (global) reputation mechanism.

We then define the weight of a link w_{ij} as (i) the number m_n of distinct *nodes* that are in the circuits from node i to j,

$$w_{ij}^{\mathrm{ccn}} = \frac{m_n}{n}, \ \ w_{ij}^{\mathrm{ccn}} \in [0, 1] \tag{8.85}$$

and (ii) the number m_e of distinct *links* that are in the circuits from node i to j,

$$w_{ij}^{\mathrm{cce}} = \frac{m_e}{n(n-1)}, \ \ w_{ij}^{\mathrm{cce}} \in [0, 1]. \tag{8.86}$$

An example of the different link weights can be seen in Fig. 8.23.

In order to compute all circuits in a directed graph G, one needs to compute the trails in G. The closed trails then are the circuits in G. We use an algorithm to compute all trails in G from a given source node s. An explanation of the algorithm is given in Appendix (B).

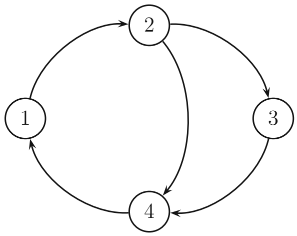


Fig. 8.24 The link e_{12} is contained in two circuits with nodes 1, 2, 4 and 1, 2, 3, 4. The number of distinct nodes in these circuits is 4 and the number of distinct links is 5. Accordingly, $w_{ij}^{\text{ccn}} = \frac{4}{4} = 1$ and $w_{ij}^{\text{cce}} = \frac{5}{12} = 0.42$

By introducing the circuit-centrality externality, we will show that more agents are permanent in the equilibrium network. The performance of the system is increased compared to the equilibrium networks that emerge with unilateral link formation without this externality.

Using circuit-centrality measure with the number of nodes (8.85), the growth of the value of knowledge is given by

$$\frac{dx_i}{dt} = -dx_i + \sum_{j=1}^{n}(ba_{ji} + b_e w_{ji}^{\text{ccn}})x_j - c\sum_{j=1}^{n} a_{ij}x_i^2. \tag{8.87}$$

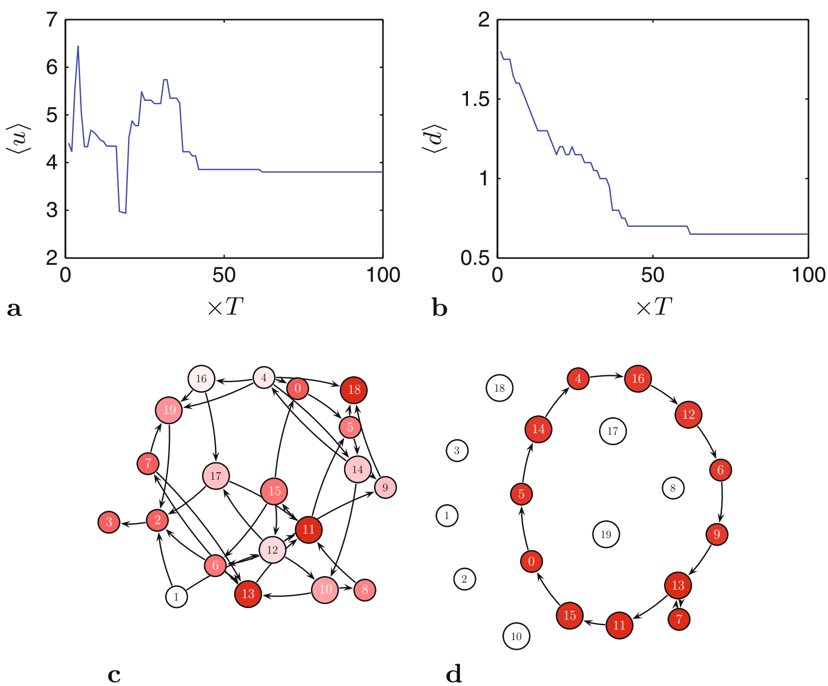


Fig. 8.25 Circuit centrality as a function of the number of nodes: (**a**) Average utility. (**b**) Average degree. (**c**) Initial random graph. (**d**) Graph after 500 iterations (in the equilibrium)

The utility is given by $u_i = \lim_{t\to\infty} x_i(t)$ (large T). Link changes are accepted on the basis of an increase in utility. Different to the centrality externality (Sect. 8.3.9.9) we observe larger cycles as the equilibrium networks. This can be seen in the simulation run in Fig. 8.25. This positive externality allows for more agents to be permanent in the equilibrium network than without an externality or with the centrality-externality. We thus obtain a higher performance of the system.

Using the circuit-centrality measure with the number of links (8.86), the growth of the value of knowledge is given by

$$\frac{dx_i}{dt} = -dx_i + \sum_{j=1}^{n}(ba_{ji} + b_e w_{ji}^{\mathrm{cce}})x_j - c\sum_{j=1}^{n} a_{ij}x_i^2. \tag{8.88}$$

The utility is given by $u_i = \lim_{t\to\infty} x_i(t)$ (large T). Link changes are accepted on the basis of an increase in utility. The circuit-centrality with the number of links values the number of interactions instead the number of agents, that take part in the transfer of knowledge. We still observe (Fig. 8.26) the emergence of circuits as equilibrium networks and a similar level of performance (in terms of the total value

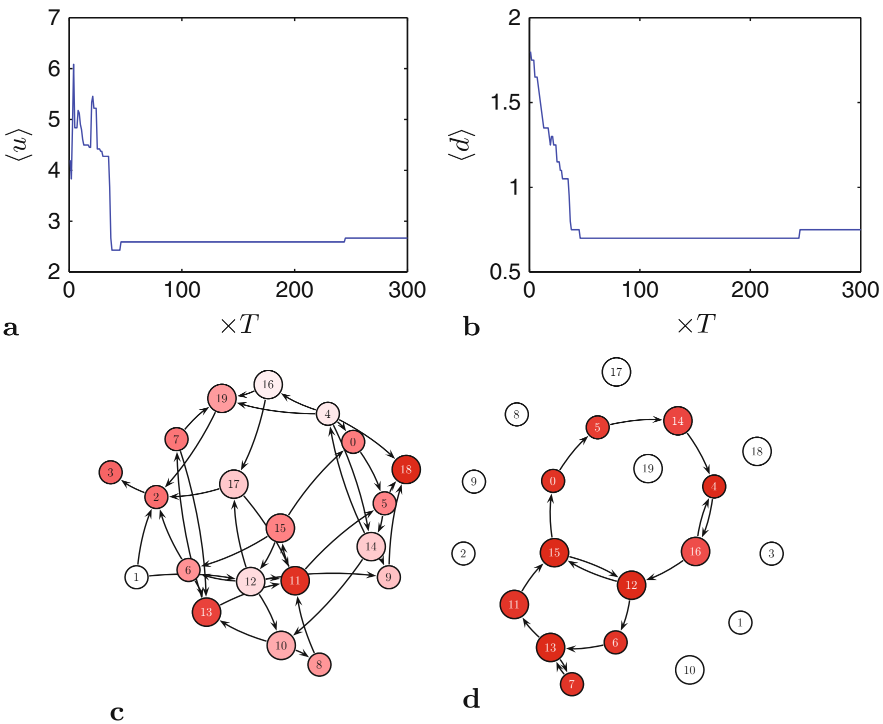


Fig. 8.26 Circuit centrality as a function of the number of links: (**a**) Average utility. (**b**) Average degree. (**c**) Initial random graph. (**d**) Graph after 500 iterations (in the equilibrium)

of knowledge of the system). But now there are more circuits (more links) in the subgraph containing the set of permanent agents. The equilibrium network has a higher level of redundancy (since its has more circuits and links) and is therefore more robust against the destruction of a single circuit (induced by a node or link failure).

8.4 Discussion and Conclusion

8.4.1 Results from the Novel Modeling Approach

In the following, we summarize the results found by studying our model of innovation dynamics, as described in Sects. 8.3. Let us start by looking at the dynamics of the value of knowledge in a static network, in Sect. (8.3.8). If we assume that growth occurs only through interaction among agents (thus neglecting "in-house" R&D capabilities), then the network sustains itself only through cycles (more precisely through closed walks or strongly connected components). Agents survive and grow only if they are part of a cycle (strongly connected component) or if they are connected to such a cycle through an incoming path.[40] We have shown that an innovation network which is acyclic will have vanishing knowledge values for all agents in the network. However, if agents form cycles they have permanent knowledge values.

Considering the evolution of the network we have studied two different settings, Extremal Dynamics and Utility Driven Dynamics. If the network evolves according to a least fit selection mechanism (extremal dynamics) then we observe crashes and recoveries of the knowledge values of the agents and the network itself. Thus, an extremal market selection mechanism which replaces the worst performing agent with a new agent cannot generate equilibrium networks nor does it sustain a high performance in the value of knowledge of the individual agents or the economy as a whole. Notice also that extremal dynamics means that agents are completely passive and have no control on whom they interact with.

In a more realistic setting (utility driven dynamics), agents decide with whom they interact and they do so in order to increase their utility. In the context of innovation, this corresponds to their value of knowledge. The information processing capabilities of agents may be limited, especially if there is a large number of agents in the economy. Thus, we allow agents to decide themselves to create or delete links on a trial and error basis. Those interactions that prove to be beneficial are maintained while detrimental ones are severed. We find that, under these conditions, the evolution of the network depends on the cost, c_{ij}, of an interaction between the agents, the type of link formation (unilateral versus bilateral), and the time horizon T after which interactions are evaluated.

[40] Jain and Krishna (2001) have denoted this set of nodes the autocatalytic set (ACS).

In the case of null cost, agents always form new links and thus the complete graph is eventually realized.

We have shown in Sect. 8.3.7 that the knowledge values of agents vanish if the underlying network does not contain a cycle (similar to the results obtained by Maxfield (1994), Rosenblatt (1957). Equilibrium networks contain cycles, a result which is similar to the findings of Bala and Goyal (2000), Kim and Wong (2007)). However, the evolution of the network driven by the selfish linking process of agents can lead to the destruction of these vital cycles. For a short time horizon T, and unilateral link formation, cycles get destroyed because agents free-ride and delete their outgoing links as it is beneficial in the short run to save the costs of supporting other agents. As a result, the whole innovation network is destroyed. On the other hand, if the time horizon is long enough, agents do not delete the cycles they are part of.

However, even when the time horizon is long, large cycles get destroyed in favor of smaller ones when agents unilaterally form or delete links. The network, starting from an initial state of high density, evolves into an absorbing state in which most of the nodes are isolated and few pairs of nodes are connected by bilateral links. These pairs are trivial cycles of length $k = 2$.[41]

Recall that pairwise connections are direct reciprocal interactions. This means that even though agents are unilaterally forming links and therefore indirect reciprocal interactions would be possible in principle (this is equivalent to interaction taking place on a cycle of length $k \geq 3$) no relation of indirect reciprocity is able to emerge nor to survive. From the point of view of the global performance of the innovation network, this is a very unsatisfactory situation.

In Sect. 8.3.9, we have studied situations in which even unilateral knowledge exchange can have a higher performance in terms of the number of permanent agents and their total value of knowledge. We introduce an externality in the knowledge growth function which increases the value of knowledge of the agent depending on their position in the network. We study a type of technology where the value of knowledge decreases with the number of agents transferring the knowledge. Here unilateral knowledge exchange still leads to equilibrium networks with a low performance and only a few permanent agents.

However, for a type of knowledge where its value increases with the number of agents that transfer and use it, more agents can be permanent in the equilibrium network and the system performance is increased. Moreover, if the number of interactions instead of the number of users determines the added value of the knowledge that is being transferred, then the equilibrium has not only a higher performance than in the setting, where knowledge is attenuated with the number of users or where no externalities are considered, but it is also more robust against node or connection failures. We observe that, in our framework, indirect reciprocity emerges

[41] This is different to the results obtained by Kim and Wong (2007) since there the benefit term in the utility of the agents depends on the size of the connected component but not on its structure.

Table 8.4 Overview of the equilibrium networks that are realized under different assumptions on the network evolution and a quadratic cost function

Network evolution / Cost function	Extremal dynamics	Utility driven dynamics		
			Unilateral	
		Bilateral	Without externality	With externality
$c_{ij} = cx_i^2$	Network breakdown	K_n	Set of C_2	Set of $C_{k \geq 2}$

if it is associated with a positive externality, taking into account the structure of the network.

If agents form or delete links bidirectionally, that means, every exchange of knowledge is direct reciprocal, the network evolves into a complete graph. This equilibrium network has a high performance and all agents are permanent. In our study, we find that unilateral knowledge exchange is always inferior to bilateral knowledge exchange. But the above discussion has shown that, when bilateral knowledge exchange is not possible and agents are sharing their knowledge unilaterally, innovation networks are still able to emerge.

The different cases studied in this section have shown that the equilibrium innovation network that is realized in the evolution of the system depends critically on the assumptions made on the behavior of agents (extremal dynamics versus utility driven dynamics), on their time horizon for evaluating their decisions and on the cost associated with the sharing of knowledge. These results are summarized in Table 8.4

8.4.2 General Conclusions

In this chapter, we studied a variety of different models for innovation networks. We started by discussing the importance of networks in economics and emphasized that these networks are intrinsically dynamic and composed of heterogeneous units. The notion of a *complex network* was used in the beginning to briefly explain how statistical physics can be involved to study them. We tried to classify different approaches to modeling economic networks, in particular we considered the connection between the state variables associated with the nodes of a network, e.g., the productivity level of a firm, and the dynamics of the network itself, i.e., the interactions between firms.

Before developing our own modeling framework, we discussed some basic models of economic networks with agents engaged in knowledge production. These models show that the economy can evolve into equilibrium networks which are not necessarily efficient. Moreover, the equilibrium networks that emerge in these models are rather simple. We briefly introduced some models in which more complicated network structures emerge, which may be closer to real-world innovation networks. We then discussed models in which cycles, i.e., closed feedback loops, play an important role in the network formation and the performance of the system (similar to Rosenblatt, 1957).

The major part of the chapter was devoted to the development of our own modeling framework, which is based on catalytic knowledge interactions. In this setting, there are permanent agents (with non-vanishing knowledge values) only if the underlying network contains a cycle. We investigated the evolution and performance of the system under different selection mechanisms, i.e., a least fit selection mechanism, denoted by Extremal Dynamics, versus Utility Driven Dynamics in which agents decide upon their interaction partners in a trial and error procedure. We observe that a least fit mechanism cannot generate stable networks nor sustain high performance in knowledge production. Moreover, such a mechanism assumes that agents are completely passive entities. In the case of Utility Driven Dynamics, agents choose their actions in order to increase their utility but their information processing capabilities are limited. If agents are evaluating their interactions after a time long enough, we obtain equilibrium networks with non-vanishing (permanent) knowledge production.

In our framework, we investigated different assumptions about the behavior of agents, that is, we either assume that agents share knowledge bilaterally or unilaterally. If all interactions are bilateral, the equilibrium network is a complete graph and it has the highest performance. However, if direct reciprocal interactions cannot be enforced (which means that links are not necessarily bilateral), we still observe the emergence of networks of knowledge sharing agents. But in the equilibrium network only bilateral interactions remain. Moreover, only a few agents are permanent and the system has a low performance compared to the case of purely bilateral interactions. However, for unilateral interactions, the number of permanent agents can be significantly increased, for a type of technology where the number of users increases its value.

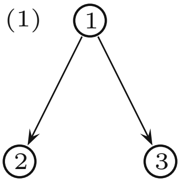
Our studies show that the range of innovation networks that can emerge in this general framework is affected by various parameters. Amongst these are information processing capabilities of agents, their time horizon, their behavior in interaction with others, the cost associated with the sharing of knowledge, and the type of technology which agents produce and transfer.

The variety of possible networks is quite large and the network model appropriate for a given application should be determined based on the specificities of the problem under investigation.

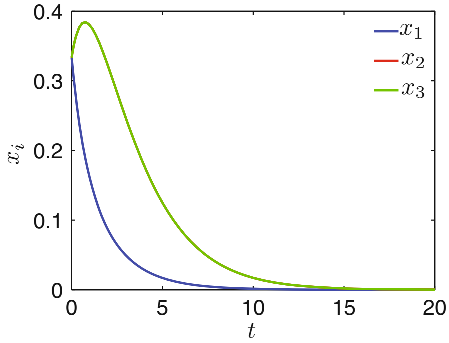
Appendix A: Stationary Solutions for Three Agents

Example 29 We compute the fixed points for all graphs (auto-morphisms) with $n = 3$ nodes and initial values $\mathbf{x}(0) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$. For the numerical integration we set $d = 0.5$, $c = 0.5$, and $b = 1$. The fixed points (stationary solutions) are denoted x_i^* for $i = 1, 2, 3$. Where possible, we give the analytical solutions for the positive fixed points. $x_i^* = 0$ is a fixed point for all graphs.

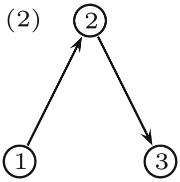
(1)



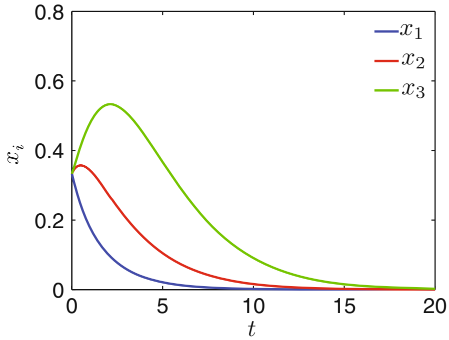
$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



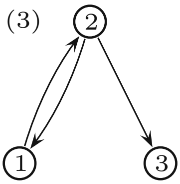
(2)



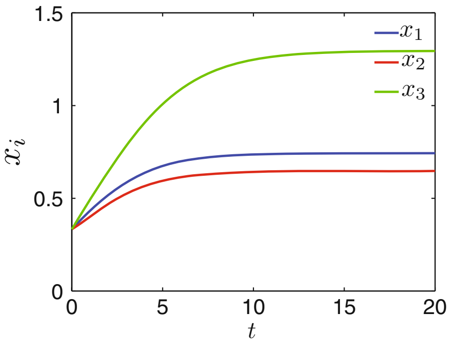
$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



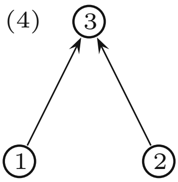
(3)



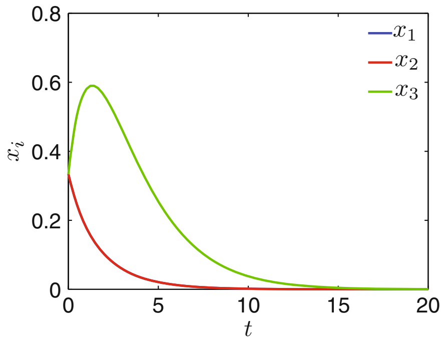
$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



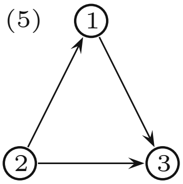
(4)



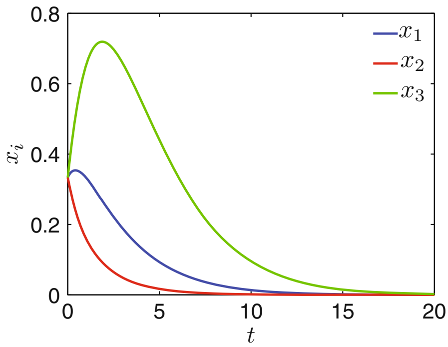
$$A_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



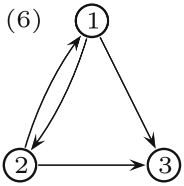
(5)



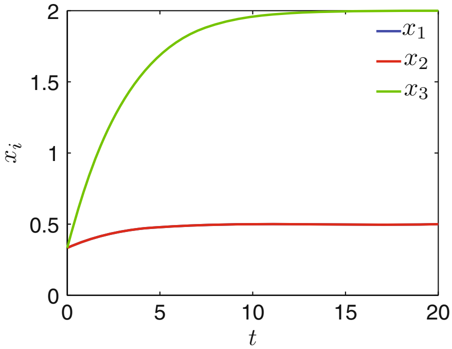
$$A_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



(6)



$$A_6 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

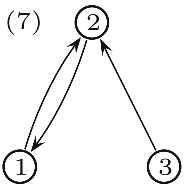


Non-trivial fixed points are given by

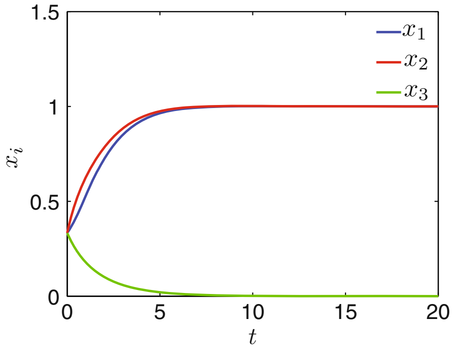
$$x_1^* = x_2^* = \frac{b-d}{2c}, \tag{8.89}$$

$$x_3^* = \frac{b(b-d)}{cd}. \tag{8.90}$$

(7)



$$A_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

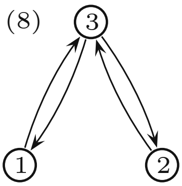


Non-trivial fixed points are given by

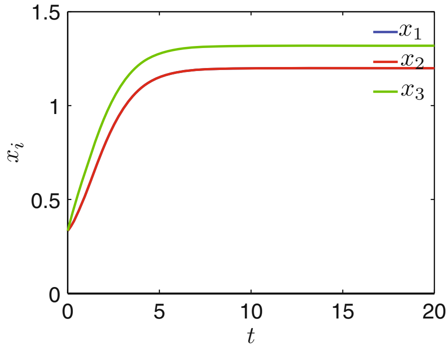
$$x_1^* = x_2^* = \frac{b-d}{c}, \tag{8.91}$$

$$x_3^* = 0. \tag{8.92}$$

(8)



$$A_8 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

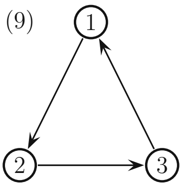


$x_1^* = x_2^*$ are the roots of the polynomial

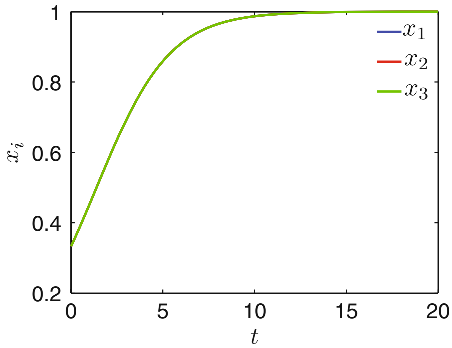
$$x^3 + \frac{2d}{c}x^2 + \frac{d(b+2d)}{2c^2}x + \frac{b(d^2-2b^2)}{2c^3} = 0$$

and $x_3^* = \frac{d}{b}x + \frac{c}{b}x^2$.

(9)



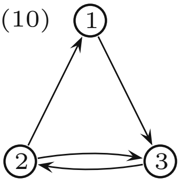
$$A_9 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



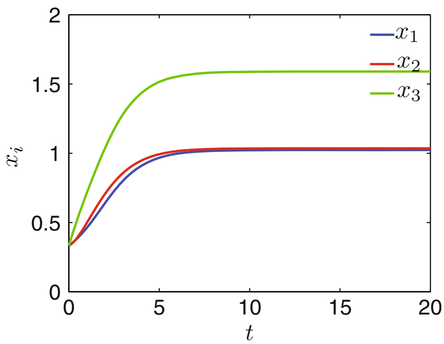
Non-trivial fixed points are given by

$$x_1^* = x_2^* = x_3^* = \frac{b-d}{c}. \tag{8.93}$$

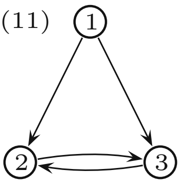
(10)



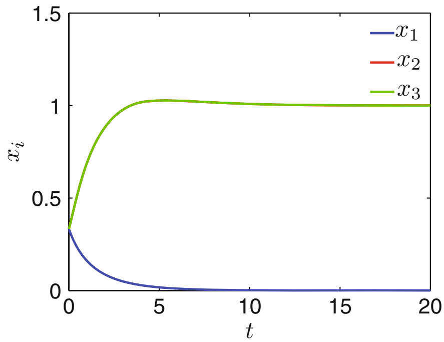
$$A_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



(11)



$$A_{11} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

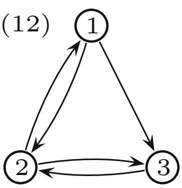


Non-trivial fixed points are given by

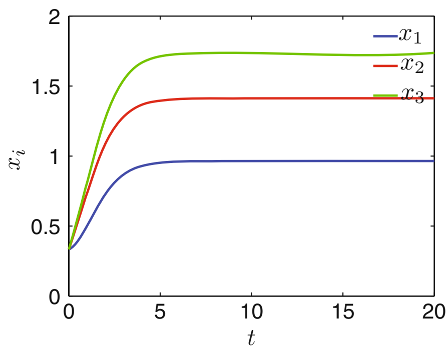
$$x_2^* = x_3^* = \frac{b-d}{c}, \tag{8.94}$$

$$x_1^* = 0. \tag{8.95}$$

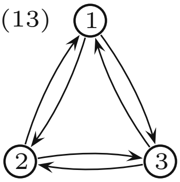
(12)



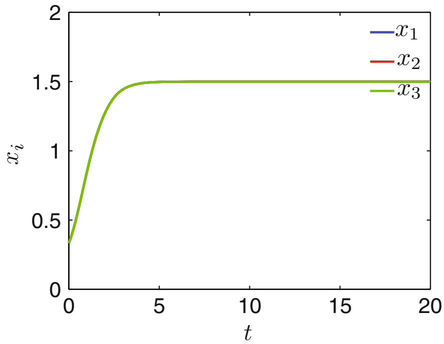
$$A_{12} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



(13)



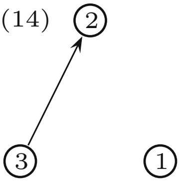
$$A_{13} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$



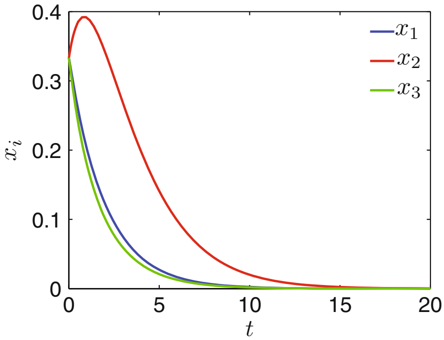
Non-trivial fixed points are given by

$$x_1^* = x_2^* = x_3^* = \frac{2b-d}{2c}. \tag{8.96}$$

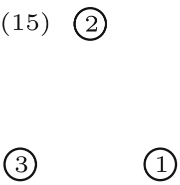
(14)



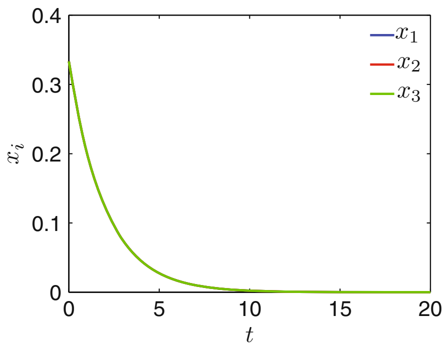
$$A_{14} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



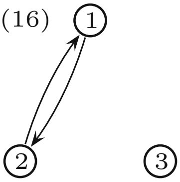
(15)



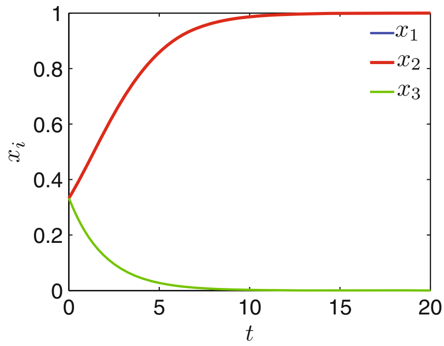
$$A_{15} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



(16)



$$A_{16} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Non-trivial fixed points are given by

$$x_3^* = 0, \tag{8.97}$$

$$x_1^* = x_2^* = \frac{b-d}{c}. \tag{8.98}$$

Appendix B: All-Trails-Single-Source Algorithm

With algorithm All-Trails-Single-Source we want to compute all trails from a given node $s \in V$ to all other nodes in a directed graph $G = (V, E)$. From these trails we can extract the trails which end in node s and thus form circuits. The pseudo-code for the All-Trails-Single-Source algorithm is given in Algorithm 1. $N^+(v)$ denotes the out-neighborhood of node v. In the following, we will give a short description of the algorithm.

Algorithm 1 All-Trails-Single-Source

```
S ← newStack();
v ← s;
W ← {}; {initialization of empty list of trails}
loop
   if ∃u ∈ N⁺(v)\{s} s.t. the link e_vu cannot be appended to W to create a new trail then
      S.push(v);
      W[u].addEdge((v, u));
      v ← u;
   else if S.isEmpty() == false then
      v ← s.pop();
   else
      break;
   end if
end loop
return W;
```

Similar to a depth-first-search algorithm, links are explored out of the most recent discovered node v that still has unexplored links leaving it. This procedure of exploring links can be represented by a search tree T. The tree T explored by Algorithm 1 contains all trails starting at the source s to every node in G.

At every node i, a list $W[i]$ of trails leading from the source s to i is assigned. When the next link e_{ij} from i to j is processed, to all trails in $W[i]$ the link e_{ij} is appended (if this is possible, meaning that no link repetition is allowed), denoted by $W[i]+e_{ij}$. At node j these trails are added, that is $W'[j] = W[j] \cup \{W[i]+e_{ij}\}$. This procedure is continued until the algorithm terminates. The algorithm terminates, if there are no further links available for exploration. The progress of Algorithm 1 on a directed graph with four nodes and two circuits containing the nodes (2, 3, 4, 1) and (2, 4, 1), respectively, is shown in Fig. 8.27.

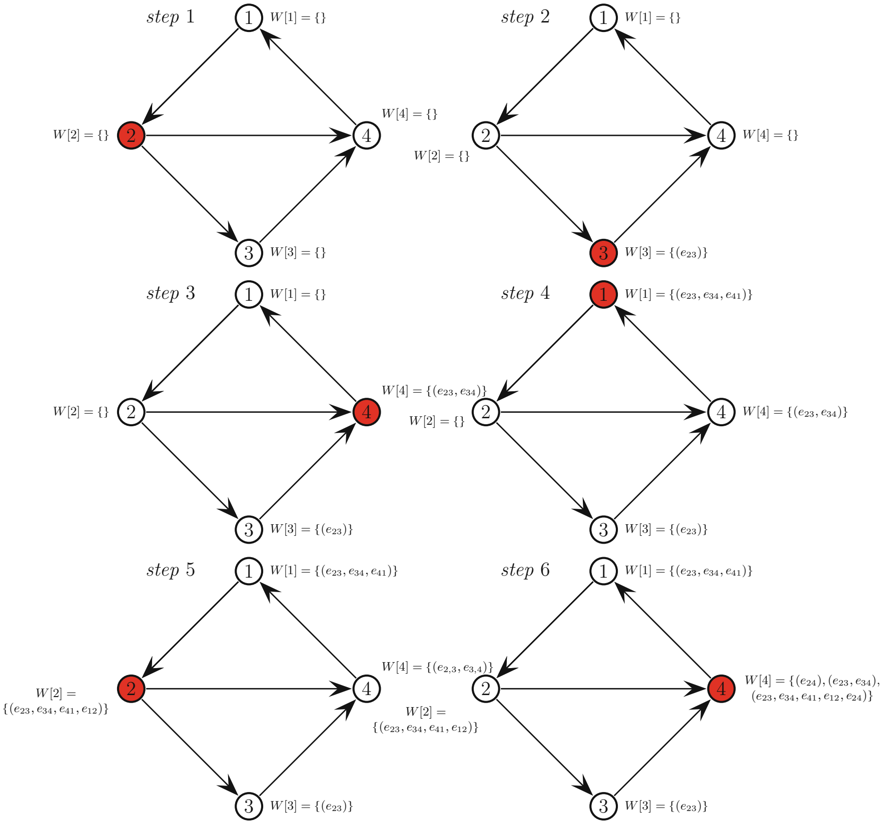


Fig. 8.27 The progress of Algorithm 1 on a directed graph with node 2 as the source node. The node indicated in red is visited in the succeeding steps. After step 6 no further trails are added to the list of trails

Appendix C: Simulation Parameters

The parameters shown in Table 8.5 have been used for the simulation runs presented in Sect. 8.3.9.6.

Table 8.5 Simulation parameters

Description	Variable	Value
Initial link creation probability	p	0.1
Initial value of knowledge	$\mathbf{x}(0)$	1.0
Number of agents (without externality)	n	30
Number of agents (with externality)	n	20
Max. numerical integration time (time horizon)	T	100
Numerical integration time step	Δt	0.05
Max. number of network updates	N	[100, 5000]
Benefit	b	0.5
Decay	d	0.5
Cost	c	0.1

Acknowledgments We are much obliged to Koen Frenken who helped a lot to improve the model presented in this chapter. Moreover, we would like to thank Giorgio Fagiolo for the clarifying discussions. In writing the chapter we have always been conscious of our debt to colleagues who have helped us in bringing this work to the present state. Among these are in particular Kerstin Press and Mauro Napoletano who have pointed us to some inconsistencies and sections which needed a more extended explanation of the material treated there in the early versions of this chapter. The discussion with Dimo Brockhoff on the algorithms were of great help. We would like to thank the discussant and the critical audience at the First International Conference on Economic Sciences with Heterogeneous Interacting Agents in Bologna, 2006. Finally, we would like to thank Peter Howitt, Robert Axtell, Herbert Gintis, and Leigh Tesfatsion for taking so much time in discussing the model at the Seventh Trento Summer School on Agent-Based Computational Economics in Trento, 2006 and Axel Leijonhufvud for making these discussions possible.

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