Decelerating Microdynamics Can Accelerate Macrodynamics in the Voter Model

Hans-Ulrich Stark, Claudio J. Tessone, and Frank Schweitzer
Chair of Systems Design, ETH Zurich, CH-8032 Zurich, Switzerland
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For the voter model, we study the effect of a memory-dependent transition rate. We assume that the transition of a spin into the opposite state decreases with the time it has been in its current state. Counterintuitively, we find that the time to reach a macroscopically ordered state can be accelerated by slowing down the microscopic dynamics in this way. This holds for different network topologies, including fully connected ones. We find that the ordering dynamics is governed by two competing processes which either stabilize the majority or the minority state. If the first one dominates, it accelerates the ordering of the system. The conclusions of this Letter are not restricted to the voter model, but remain valid to many other spin systems as well.

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How fast an out-of-equilibrium system reaches an ordered state has been a central question in statistical physics, but also in disciplines such as chemistry, biology, and social sciences. Despite its simple structure, the ‘‘voter model’’ has served as a paradigm to study this question [1]. It is one of the few spin systems that can be analytically solved in regular lattices [2]. In physics, it was investigated how the time to reach the equilibrium state depends on the system size, the initial configuration, and the topology of the interactions [3]. Among its prominent properties, the magnetization conservation has been studied extensively [4]. Furthermore, the formation and growth of state domains was studied, showing the existence of coarsening without surface tension in two-dimensional systems [5]. The voter model also found numerous interdisciplinary applications, e.g., in chemical kinetics [6] and in ecological [7,8] and social systems [9]. Its properties have also served to complete the understanding of other spin systems, such as the Ising model and spin glasses [10].

To assume that transition rates are constant in time is (in general) not valid for nonequilibrium systems. A good example are spin glasses, where the effective temperature of the system changes with the time elapsed since a given perturbation was applied [11]. In this Letter, we consider that, for each site, the transition rates are not constant, but *decrease* with the time elapsed since the last change of state (namely, its *persistence time*). We refer to this change as *increasing inertia*. The level of inertia is measured by how fast the transition rates decrease with persistence time. Dependent on the context of the voter model, this mechanism has different interpretations. In a social context, the longer a voter already stays with its current state, the less it may be inclined to change it in the next time step, which can be interpreted as conviction. In models of species competition [7], this would imply that neighboring species are less likely to be displaced at a later stage of growth.

Obviously, increasing inertia leads to slower microscopic dynamics. Against intuition and in contrast to results with fixed (homogeneous or heterogeneous) values of inertia, we find that the time to reach an ordered state can be effectively reduced. We further find that this phenomenon exists independently of the exact network topology in which the system is embedded. We show that the unexpected reduction of the time to reach an ordered state is related to the break of magnetization conservation, which holds for the standard voter model. This break originates from the evolving heterogeneity in the transition probabilities within the voter population, which, in the extended model, depends on the distribution of the persistence times.

The voter model denotes a simple binary system composed of N *voters*, each of which can be in one of two states (often referred to as *opinions*), $\sigma_i = \pm 1$. A voter is selected at random and adopts the state of a randomly chosen neighbor. After N such update events, time is increased by 1. In this work, we consider homogeneous networks, where all voters have the same number of neighbors. In the standard voter model, the transition rate at which voter i switches to the opposite state, $\omega^V(-\sigma_i|\sigma_i)$, is proportional to the frequency of state $-\sigma_i$ in $\{i\}$, the set of the k neighbors of i, namely

$$\omega^V(-\sigma_i|\sigma_i) = \frac{\beta}{2}\left(1 - \frac{\sigma_i}{k}\sum_{j\in\{i\}}\sigma_j\right). \tag{1}$$

The prefactor β determines the time scale of the transitions and is set to $\beta = 1$. In order to describe the dynamics on the macrolevel, we introduce the global densities of voters with state $+1$ as $A(t)$ and with state -1 as $B(t)$. The instantaneous magnetization is then given by $M(t) = A(t) - B(t)$. Starting from a random distribution of states, we have $M(0) = 0$. The emergence of a completely ordered state (which is often referred to as *consensus*), is characterized by $|M| = 1$. The time to reach consensus, T_κ, is obtained through an average over many realizations. The dynamics of the global frequencies is formally given by the rate equation

$$\dot{A}(t) = -\dot{B}(t) = \Omega^V(+1|-1)B(t) - \Omega^V(-1|+1)A(t).$$

The *macroscopic* transition rates Ω^V have to be obtained from the aggregation of the microscopic dynamics given by Eq. (1). A simple expression for these can be found in the mean-field limit. There, it is assumed that the frequencies of states in the local neighborhood can be replaced by the global ones. This gives $\Omega^V(+1|-1) = A(t)$, $\Omega^V(-1|+1) = B(t)$ and leads to $\dot{A}(t) = A(t)B(t) - B(t)A(t) \equiv 0$. For an ensemble average, the frequency of the outcome of a particular consensus state $+1$ is equal to the initial frequency $A(0)$ of state $+1$, which implies the conservation of magnetization. It is worth noticing that, for a single realization, the dynamics of the voter model is a fluctuation driven process that, for finite system sizes, always reaches consensus towards either $+1$ or -1. We now investigate how this dynamics changes if we modify the voter model by assuming that voters additionally have an inertia $\nu_i \in [0, 1]$ which leads to a decrease of the transition rate to change their state

$$\omega(-\sigma_i|\sigma_i, \nu_i) = (1 - \nu_i)\omega^V(-\sigma_i|\sigma_i). \tag{2}$$

Obviously, if all voters have the same fixed value of inertia $\nu_\bullet$, the dynamics is equivalent to the standard voter model with the time scaled by a factor $(1 - \nu_\bullet)^{-1}$. Similar results are obtained if the inertia values are randomly distributed in the system: higher consensus times are found for increasing levels of inertia. In our model, however, we consider an individual and evolving inertia ν_i that depends on the persistence time τ_i the voter has been keeping its current state. For the sake of simplicity, the results presented here assume that the individual inertia ν_i increases linearly with persistence time τ_i, μ being the "strength" of this response, until it reaches a saturation value ν_s, i.e., $\nu(\tau_i) = \min[\mu\tau_i, \nu_s]$. Choosing $\nu_s < 1$ avoids trivial frozen states of the dynamics [12]. The rate of inertia growth μ determines the number of time steps until the maximal inertia value is reached, denoted as $\tau_s = [\nu_s/\mu]$.

Increasing μ increases the level of inertia within the voter population, thereby slowing down the microscopic dynamics. As in the case with fixed inertia, one would intuitively assume an increase of the average time to reach consensus. Interestingly, this is not always the case as simulation results of $T_\kappa(\mu)$ show for different network topologies (see Fig. 1). Instead, it is found that there is an intermediate value μ^*, which leads to a global minimum in T_κ [13]. For $\mu < \mu^*$, consensus times decrease with increasing μ values. Only for $\mu > \mu^*$, higher levels of inertia result in increasing consensus times.

For a two-dimensional lattice, shown in Fig. 1(a), we find $\mu^* \propto 1/\ln N$. Simulations of regular lattices in other dimensions show that the nonmonotonic effect on the consensus times is amplified in higher dimensionality of the system. Being barely noticeable for $d = 1$, the ratio between $T_\kappa(\mu^*)$ and $T_\kappa(\mu = 0)$ (i.e., the standard voter model) decreases for $d = 3$ and $d = 4$. We further compare the scaling of T_κ with system size N for the standard and the modified voter model. The first one gives for one-dimensional regular lattices ($d = 1$) $T_\kappa \propto N^2$ and for two-dimensional regular lattices ($d = 2$) $T_\kappa \propto N \log N$. For $d > 2$ the system does not always reach an ordered state in the thermodynamic limit. In finite systems, however, one finds $T_\kappa \sim N$. In the modified voter model, we instead find that $T_\kappa(\mu^*)$ scales with system size as a power law, $T_\kappa(\mu^*) \propto N^\alpha$ [see inset in Fig. 1(a)], where $\alpha = 1.99 \pm 0.14$ for $d = 1$ (i.e., in agreement with the standard voter model), $\alpha = 0.98 \pm 0.04$ for $d = 2$, $\alpha = 0.5 \pm 0.08$ for $d = 3$, and $\alpha = 0.3 \pm 0.03$ for $d = 4$. For fixed values of $\mu > \mu^*$, the same scalings apply.

To cope with the network topology, in Fig. 1(b) we plot the dependence of the consensus times T_κ for small-world networks built with different rewiring probabilities. The

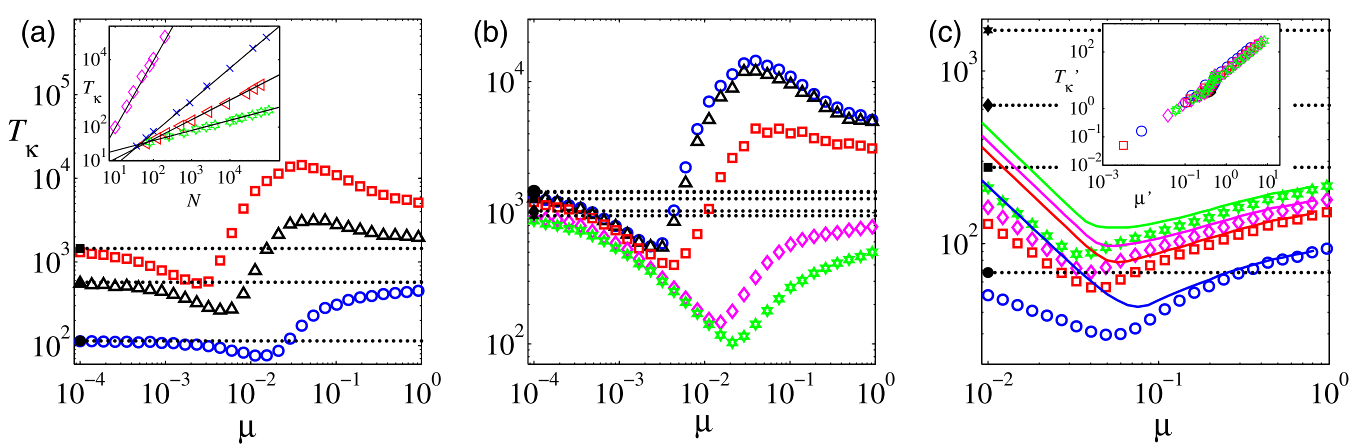

FIG. 1 (color online). Average consensus times T_κ for varying values of the inertia slope μ and fixed saturation value $\nu_s = 0.9$. Sample sizes vary between 10^3–10^4 simulation runs. Filled, black symbols always indicate the values of T_κ at $\mu = 0$. (a) $2d$ regular lattices ($k_i = 4$) with system sizes: (○) $N = 100$, (△) $N = 400$, (□) $N = 900$. The inset shows how consensus time scales with system size in regular lattices at $\mu = \mu^*$: (◇) $1d$, (×) $2d$, (◁) $3d$, (★) $4d$. (b) Small-world networks obtained by randomly rewiring a $2d$ regular lattice with probability: (○) $p_r = 0$, (△) $p_r = 0.001$, (□) $p_r = 0.01$, (◇) $p_r = 0.1$, (★) $p_r = 1$. The system size is $N = 900$. (c) Fully connected networks (mean-field case, $k_i = N - 1$) with system sizes: (○) $N = 100$, (□) $N = 900$, (◇) $N = 2500$, (★) $N = 10^4$. Lines represent the numerical solutions of Eqs. (5)–(7) with the specifications in the text. The inset shows the collapse of the simulation curves by scaling μ and T_κ as explained in the text.

degree of each node is kept constant by randomly selecting a pair of edges and exchanging their ends with probability p [14]. It can be seen that the effect of reduced consensus times for intermediate values of μ still exist and are amplified by increasing the randomness of the network. This result implies that the spatial extension of the system, e.g., in regular lattices, does not play a crucial role in the emergence of this phenomenon. This can be confirmed by investigating the case shown in Fig. 1(c), in which the neighborhood network is a fully connected one (the solid lines correspond to a theoretical approximation introduced below). The inset shows the results of a scaling analysis, exhibiting the collapse of all the curves by applying the scaling relations $\mu' = |\mu \ln(\eta N) - \mu_1|$, and $T'_\kappa = T_\kappa / \ln(N/\xi)\mu'$, with $\eta = 1.8(1)$, $\mu_1 = 1.5(1)$, $\xi = 7.5(1)$. This shows that the location of the minimum, as well as T_κ, scales logarithmically with N.

The fact of reaching a final state faster by decelerating the dynamics microscopically has some resemblances with the "slower-is-faster" effect discovered in panic research [15]. However, the origin of the phenomenon discussed here is quite different, as we can demonstrate by the following analytical approach. First, note that voters are fully characterized by their current state ± 1 and their persistence time τ. Thus, we introduce the global frequencies $a_\tau(t)$, $b_\tau(t)$ for subpopulations of voters with state $+1$, -1 (respectively) and persistence time τ. Thus, these frequencies satisfy

$$A(t) = \sum_\tau a_\tau(t), \qquad B(t) = \sum_\tau b_\tau(t). \tag{3}$$

Formally, the rate equations for the evolution of these subpopulations in the mean-field limit are given by

$$\dot{a}_\tau(t) = \sum_{\tau'} [\Omega(a_\tau | a_{\tau'}) a_{\tau'} + \Omega(a_\tau | b_{\tau'}) b_{\tau'}] - \sum_{\tau'} [\Omega(a_{\tau'} | a_\tau) + \Omega(b_{\tau'} | a_\tau)] a_\tau. \tag{4}$$

Because of symmetry, the expressions for $\dot{b}_\tau(t)$ are obtained by consistently exchanging $A \leftrightarrow B$ and $a_\tau \leftrightarrow b_\tau$.

Note that most of the terms in Eq. (4) vanish because for a voter only two transitions are possible: (i) it changes its state, thereby resetting its τ to zero, or (ii) it keeps its current state and increases its persistence time by one. Case (i) is associated with the transition rate $\Omega(b_0|a_\tau)$, that in the mean-field limit reads $\Omega(b_0|a_\tau) = (1 - \nu(\tau))B(t)$. $B(t)$ is the frequency of voters with the opposite state that trigger this transition, while the prefactor $(1 - \nu(\tau))$ is due to the inertia of voters of class a_τ to change their state. For case (ii), $\Omega(a_{\tau+1}|a_\tau) = 1 - \Omega(b_0|a_\tau)$, since no voter can remain in the same subpopulation. I.e., in the mean-field limit, the corresponding transition rates are $\Omega(a_{\tau+1}|a_\tau) = A(t) + \nu(\tau)B(t)$. Therefore, if $\tau > 0$, Eq. (4) reduces to

$$\dot{a}_\tau(t) = \Omega(a_\tau | a_{\tau-1}) a_{\tau-1}(t) - a_\tau(t) = [A(t) + \nu(\tau - 1)B(t)] a_{\tau-1}(t) - a_\tau(t). \tag{5}$$

On the other hand, voters with $\tau = 0$ evolve as

$$\dot{a}_0(t) = \sum_\tau \Omega_b(a_0 | b_\tau) b_\tau(t) - a_0(t) = A(t)[B(t) - I_B(t)] - a_0(t). \tag{6}$$

Because of the linear dependence of the transition rates on inertia, the terms involving ν can be merged into $I_B(t)$ and $I_A(t)$, namely, the average inertia of voters with state -1 and $+1$, respectively, i.e.,

$$I_A(t) = \sum_\tau \nu(\tau) a_\tau(t); \qquad I_B(t) = \sum_\tau \nu(\tau) b_\tau(t). \tag{7}$$

Expressions (5)–(7) and the corresponding ones for subpopulations b_τ can be used to give an estimate of the time to reach consensus in the mean-field limit. Let us consider an initial state $a_0(t) = A(0) = 1/2 + N^{-1}$ and $b_0(t) = B(0) = 1/2 - N^{-1}$; i.e., voters with state $+1$ are in a slight majority. By neglecting fluctuations in the frequencies (which drive the dynamics in the standard voter model), these equations are iterated until $B(t') < N^{-1}$ (i.e., for a system size N, if the frequency of the minority state falls below N^{-1}, the absorbing state is reached). Then, we assume $T_\kappa = t'$. The full lines in Fig. 1(c) show the results of this theoretical approach, exhibiting the minimum and displaying good agreement with the simulation results for large values of μ. For low values of μ, fluctuations drive the system *faster* into consensus compared to the deterministic approach.

Inserting Eqs. (5) and (6) into the time-derivative of Eq. (3) yields, after some straightforward algebra, the time evolution of the global frequencies

$$\dot{A}(t) = I_A(t)B(t) - I_B(t)A(t). \tag{8}$$

Remarkably, the magnetization conservation is now broken because of the influence of the evolving inertia in the two possible states. For $\nu(\tau) = \nu_\bullet$ (that includes the standard voter model, $\nu_\bullet = 0$), we regain the magnetization conservation. Interestingly enough, Eq. (8) implies that the frequency $A(t)$ grows iff $I_A(t)/A(t) > I_B(t)/B(t)$.

When the time dependence of the inertia on the persistence time is a linear one, as assumed in this Letter, inserting Eqs. (5) and (6) into Eq. (7) we obtain an equation for the time evolution of $I_A(t)$ up to first order in μ:

$$\dot{I}_A(t) = A(t)I_A(t) + \mu A^2(t) - I_A(t) + \mathcal{O}(\mu^2, a_T). \tag{9}$$

Here, $a_T = \sum_{\tau \geq \tau_s} a_\tau$ contains all subpopulations with maximum inertia. Equations (8) and (9) correspond to a macroscopic level description of this model. This system of equations has a saddle point, $A = B = 1/2$, $I_A = I_B = \mu/2 + \mathcal{O}(\mu^2)$, and two stable fixed points, one at $A = 1$, $I_A = \nu_s$ and another at $B = 1$, $I_B = \nu_s$. Note that the saddle point is close to the initial condition of the simulations. Neglecting fluctuations, the time to reach consensus has two main contributions: (i) the time to escape from the saddle point T_s; and (ii) the time to reach the stable fixed point T_f; namely $T_\kappa \sim T_s + T_f$. We then linearize the system around the fixed points and calculate the largest eigenvalues λ_s and λ_f (for the saddle and the stable fixed points, respectively) as a function of μ. A simple argument shows that $T_{s,f} \sim \ln N/|\lambda_{s,f}(\mu)|$. At the saddle point, we find $\lambda_s(\mu) = \sqrt{1 + 20\mu + 4\mu^2} - 2\mu - 1 + \mathcal{O}(\mu^2)$, which equals 0 at $\mu = 0$ and monotonically increases with μ. For larger values of μ, where the first order term expansion is no longer valid, numerical computations show that λ_s continues to increase monotonically with μ. This means that for larger inertia growth rates μ, the system will escape faster from the saddle point, thereby reducing the contribution T_s to the consensus time T_κ. On the other hand, for $\mu \to 0$, λ_s vanishes and the system leaves the saddle point only due to fluctuations.

Near the stable fixed points the contribution of a_T to Eq. (9) cannot be neglected anymore. We then obtain $\lambda_{f,1} = -\nu_s$ for $\mu < 1 - \nu_s$, while $\lambda_{f,2} = \mu - 1$ for $\mu \geq 1 - \nu_s$. Interestingly, both reflect different processes: the eigenvalue $\lambda_{f,1}$ is connected to voters sharing the majority state which are, at the level of ν_s, inertial to adopt the minority one (signaled by $\lambda_{f,1}$ being constant). For $\mu \geq 1 - \nu_s$, the largest eigenvalue $\lambda_{f,2}$ is related to voters with the minority state that are, for increasing μ, more inertial to adopt the majority state (apparent by the decrease in $|\lambda_{f,2}|$).

The contributions T_s and T_f are two competing factors in the dynamics towards consensus. Qualitatively, they can be understood as follows: in the beginning of the dynamics, the inertia mechanism amplifies any small asymmetry in the initial conditions. While this causes faster time to consensus for (small) increasing values of μ, for sufficiently large values of inertia growth, another process outweighs the former: the rate of minority voters converting to the final consensus state is considerably reduced, too. It is worth mentioning that the phenomenon described here is robust against changes in the initial condition: starting from $I_A = I_B < \nu_s$, it holds for any initial frequencies of opinions. Conversely, starting from $A = B = 1/2$, it holds for any $I_A \neq I_B$.

Summarizing, we investigated the role of microscopically time-dependent transition rates. In particular, we consider that the microscopic transition rates decrease with the time elapsed since the last state change of a given site (called *inertia*). Counterintuitively, we find that intermediate inertia values may lead to much lower times to reach the absorbing state, i.e., an accelerated dynamics. It is important to emphasize that this final state is not an arbitrary one, but most interestingly, it is always the *ordered* one. The mechanism behind this phenomenon is the existence of two competing processes near the initial condition and absorbing states. Because of the general analytical approach taken in this Letter, we emphasize that this phenomenon is not restricted to the voter model, but is expected to appear near the absorbing states of any spin system, whenever the inertia mechanism is present.

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