Dynamics of Companies

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Thanks to …
L. Amaral, H. Takayasu, S. Jain, S. Battiston, B. Drossel
Topics of Interest

- different perspectives:
  - economic: theory of the firm, ...
  - management: strategies for entrepreneurship, ...
  - production: supply chains, ...
- physics perspective: collective effects
  - ensembles of companies: \( i = 1, \ldots, N \)
  - simple characterization: company “size” \( x_i(t) \)
    income, output, employees, ...
- aggregated variables
- our focus:
  - growth of companies \( \Rightarrow \) size distribution
  - interaction of companies \( \Rightarrow \) network structure
  - structure of companies \( \Rightarrow \) hierarchies
  - decisions in companies \( \Rightarrow \) opinion formation
Dynamics of Companies: Firm Model Tree of its Research Evolution
Company Growth

- set of companies: $i = 1, \ldots, N$
  - $x_i(t)$: company “size”, growth rate: $dx_i/dt = \mathcal{F}_i(t)$
- $\mathcal{F}_i(t)$ with $\langle \mathcal{F}_i(t) \rangle = 0$, $\langle \mathcal{F}_i(t)\mathcal{F}_i(t') \rangle = S\delta_{ij}\delta(t - t')$

\[ x_i(t + \Delta t) = x_i(t) + \sqrt{S\Delta t} \xi_i \]

- growth as random walk (Bachelier, 1900)
- $\mathcal{F}_i = f(x_i) + \ldots \Rightarrow$
  - independent growth, proportional to size (Gibrat, 1931)
- $\mathcal{F}_i = f(x_j, x_k) + \ldots$
  - growth through innovation networks
Gibrat’s Model

\[ \dot{x}_i = F_i = f(x_i) + \ldots = b_i x_i \]

- “Law of proportionate growth” (Gibrat ’30, ’31; Sutton ’97)
- no interactions between firms

\[ x_i(t + \Delta t) = x_i(t) [1 + b_i(t)] \]

Assumptions:
- \( b_i(t) \): independent of \( i \), no temporal correlations (random noise)
- \( t \gg \Delta t \):
  \[ x(t) = x(0)(1 + b(1))(1 + b(2)) \cdots (1 + b(t)) \]
- growth “rates”:
  \( R(t) = x(t+1)/x(t), \ t \gg \Delta t, \ \ln(1 + b) \approx b \)

\[ \ln R(t) = \sum_{n=1}^{t} b(n) \]

⇒ random walk for \( \ln R(t) \) ⇒ log-normal distribution for \( x_i(t) \)
Log-normal vs Power Law Distribution

\[ x_{t+1} = \lambda_t x_t \quad \text{with} \quad \lambda = b + 1 \]

\[
P(x_t) = \frac{1}{\sqrt{2\pi Dt x_t}} \exp\left[ -\frac{1}{2Dt} (\log x_t - vt)^2 \right]
\]

\[ \nu = \langle \log \lambda \rangle ; \quad D = \langle (\log \lambda)^2 \rangle - \langle \log \lambda \rangle^2 \]

rewriting:

\[
P(x_t) = \frac{1}{\sqrt{2\pi Dt}} \frac{1}{x_{t}^{1+\mu(x_t)vt}} e^{\mu(x_t)vt} ; \quad \mu(x_t) = \frac{1}{2Dt} \log \frac{x_t}{e^{vt}}
\]

\[ x_t \ll e^{(\nu+2D)t} \quad \text{yields} \quad \mu(x_t) \ll 1 \]

\[ \Rightarrow \text{log-normal and } 1/x_t \text{ undistinguishable} \]

\[ x_t \gg e^{(\nu+2D)t} \Rightarrow \mu(x_t) \rightarrow \infty (!!) \]
Empirical Evidence?


![Graph showing the log-normal distribution of company sizes](http://www.sg.ethz.ch/)
Empirical distribution of growth rates (Amaral et al, 1997) ⇒ depend on size, tent-shape exponential distribution
Empirical distribution of standard deviation of growth rates (Amaral et al, 1997) ⇒ depend on size, power-law distribution
Stylized facts:

- log-normal distribution of company sizes

\[ P(x) = \frac{1}{\sqrt{2\pi} \sigma x} \exp \left[ \frac{(- \ln x - \mu)^2}{2\sigma^2} \right] \]

- exponential growth ratio distribution

\[ P(r_1 | x_0) = \frac{1}{\sqrt{2} \sigma_1(x_0)} \exp \left[ - \frac{\sqrt{2} | r_1 - \bar{r}_1(x_0)|}{\sigma_1(x_0)} \right] \]

- power-law distribution of the standard deviations

\[ \sigma_1(x_0) \sim x_0^{-\beta} ; \quad \beta < 0.5 \]
Explanations

- correlations in the growth rates
- company is attracted to an “optimal size”

\[
\frac{x_{t+\Delta t}}{x_t} = \begin{cases} 
ke^{\varepsilon_t}, & x_t < x^* \\
\frac{1}{k}e^{\varepsilon_t}, & x_t > x^*, 
\end{cases}
\]

- growth depends on properties of management hierarchies
- \(n\) levels, \(z\) mean branching ratio, decisions on higher level are followed with prob \(\pi\)

\[
\beta = \begin{cases} 
-\ln(\pi)/\ln(z), & \text{if } \pi > z^{-1/2} \\
1/2, & \text{if } \pi < z^{-1/2}
\end{cases}
\]

- \(\beta\) decreases in time \(\Leftrightarrow\) companies better coordinated
Micromodel of company growth

- Amaral et al (2001)
  - firm consist of several subunits (divisions): $\xi(t)$
  - firm size: $S(t) \equiv \sum_i \xi_i(t)$
    - growth from independent growth of subunits
  - entry: $t = 0$: firm created with single unit of size $\xi_1(t = 0)$
  - exit: $S < S_{\text{min}}$: firm not economically viable
The evolution of a division

- random multiplicative process:

\[ \Delta \xi_i(t) \equiv \xi_i(t) \nu_i(t) \]

- \( \Delta \xi_i(t) < S_{\text{min}} \): division evolves by changing its size
  \[ \xi_i(t + 1) = \xi_i(t) + \Delta \xi_i(t) \]
  - if \( \xi_i < S_{\text{min}} \), then with probability \( p_a \) division \( i \) is absorbed by division 1

- \( \Delta \xi_i(t) > S_{\text{min}} \)
  - with probability \( p_f \) division \( i \) does not change and a new division \( j \) is created with size \( \xi_j(t + 1) = \Delta \xi_i(t) \)
  - with probability \( (1 - p_f) \) division \( i \) evolves
    \[ \xi_i(t + 1) = \xi_i(t) + \Delta \xi_i(t) , \]
Schematic representation of Amaral et al.’s model
Further Improvements of Gibrat

- **economic idea**: simple entry dynamics (Simon & Bonini ’58)
- **mathematic idea**: add more noise! (Kesten ’73)

\[ x(t + 1) = x(t)[1 + b(t)] + a(t) \]

- \( b \), a positive independent random variables
- \( a(t) \) acts as “effective repulsion” from zero (Sornette & Cont ’97)

- **practical idea**: fit parameters (Takayasu et al ’04)

\[ x(t + 1) = \alpha(t)\lambda(t, x) x(t) + a(t) \]

- \([1 + b] \rightarrow \lambda(x, t)\): growth depends on size
- estimation from \( \ln R(t) = \ln\{x(t + 1)/x(t)\} \) with standard deviation \( \sigma(x) \)
Comparison with real company data

Takayasu et al '04: income of 15,000 US and 15,000 non-US comp., 80,000 Japanese comp. (income > 40 Mio Yen), before tax.

(Takayasu et al 2004)
Company growth

\[ x(t+1) = \alpha(t) \lambda(t, x) x(t) + f(t) \]

- \( \lambda(x, t) \): growth depends on size
  - estimation from log growth rate: \( \log R(t) = \log x(t+1) - \log x(t) \)
    - with standard deviation \( \sigma(x) \)
  - for large \( x \): \( \sigma_0, f(t)/x \) negligible
  - scaling by means of normalized growth: \( R^{\sigma(x)/\sigma_0} \)

\[(Takayasu et al 2004)\]
\( \alpha(t) \) either +1 (growth) or -1 (slump)
prob. determined empirically from large \( |x(t)| \)

\[
\alpha(t) = \begin{cases} 
1 & \text{with prob. } 0.97 \ (x(t) > 0), \ 0.75 \ (x(t) < 0) \\
-1 & \text{with prob. } 0.03 \ (x(t) > 0), \ 0.25 \ (x(t) < 0)
\end{cases}
\]

(Takayasu et al 2004)
Forecast by means of Monte Carlo Simulations

- initial state: 6,000 companies, $x_i(0) = 100$
- coefficients estimated from real data
- $t = 50$: qualitative agreement with real distribution (US)
- with constant growth rate distribution: firms income will keep growing for more than 100 years

(Takayasu et al 2004)
Investment Strategies?

- normalized cumulative income for 5 years: \( I = \sum_{n=1}^{5} x(n)/x(0) \)

\[
\langle \sum I(t)/I(0) \rangle \propto I(0)^{-1}
\]

- for \( x(0) > 10^6 \)\$: \( I \propto x(0) \Rightarrow \) invest in small firms?

(Takayasu et al 2004)
small firms: large growth rates, but also large variances (notice the asymmetric distribution)

(Takayasu et al 2004)
- investment strategy: tradeoff between profits and risks
  investment efficiency: relation between $\langle I \rangle$ and $\sigma(I)$

$$E(c, x(0)) = \langle \sum \frac{x(5)}{x(0)} \rangle - c \sigma \left( \sum \frac{x(5)}{x(0)} \right)$$

(Takayasu et al 2004)
Growth through Network Effects

\[ \dot{x}_i = \mathcal{F}_i = f(x_j, x_k) + \ldots \]

\[ \frac{dx_i}{dt} = \sum_{j=1}^{N} c_{ij} x_j - \Phi x_i \quad (\text{Jain/Krishna '98, '01}) \]

- \( c_{ij} \in \{0, 1\} \Rightarrow \) represents a directed network
  - \( j \) catalyzes the growth of \( i \), link probability \( p \)
  - \( i \) is connected to \( m = p(N - 1) \) other companies (on average)

- **two time scales:**
  - company growth (fast), network dynamics (slow)

- **assumption:** extremal dynamics \( \Rightarrow \) minimum performance threshold
Questions:
- Under which conditions do companies survive?
- Which structures of innovation networks emerge?
- What happens if selection pressure is increased?

Results of computer simulations:
Emergence of a core of *cooperative* companies, and a *parasitic* periphery, considerable crashes and recovery
Network effects
Growth through network effects
Network effects

Growth through network effects
Network effects

Growth through network effects

$t=1290$
Get connected

- $N$ agents with $\theta_i \in \{-1, +1\}$; ruling opinion $\theta_G = +1$
- CEO proposal $\Rightarrow N_+$ supporters, $N_-$ objectors
- additional (outside) ties between board members $\Rightarrow$ weight $J_{ij}$

(Battiston et al 2003)
Model of board decisions (Battiston et al, 2003)

- Probability of agent $i$ to approve CEO proposal ($\theta_G = +1$):

$$p = \frac{1}{1 + \exp \left\{ -2\beta h_i(t) \right\}} ; \quad h_i(t) = \sum_{j \in NN} J_{ij} \theta_j$$

Influence of lobby: additional force

$$F \sim \sum J_{ij} \theta_i(t = 0) \Rightarrow \text{minority of well-connected members can drive the majority’s decision}$$
The importance of speaking first

Instead of random sequential update: one at a time
memory length $\gamma \Rightarrow h^*_i = (1 - \gamma) h_i + \gamma J_{ij} \theta_j$

\[(Battiston et al 2003)\]
Formation of Hierarchies

- complex system ("company"): various organizational hierarchies
  - global aim: increase of "productivity" (utility, fitness, ...)

Simple, but illustrative model of Drossel '99:

- productivity of 1 unit at (lowest) level 1: \( P_1(1) \) (negligible)
- productivity of \( N \) interacting units at (lowest) level 1: \( P_1(N) \)
  - increases with interaction possibilities: \( P \sim N(N-1) \)
  - decreases with costs of interaction (e.g. transportation costs)
    if system size increases linearly with \( N \), then \( P \sim -N(N-1)N \)

\[
P_1(N) = \left[ g_1N - c_1N^2 \right](N-1)
\]

- maximum size: \( P_1(N) \rightarrow 0: N_{\text{max}} = g_1/c_1 \)
- maximum productivity per unit: \( P_1(N)/N \rightarrow \max \)
  \( \Rightarrow \) optimal size \( N_{\text{opt}} = (g_1 + c_1)/2c_1 \)

- reasons to split into \( N = N' + N'' \) if \( P_1(N) < P_1(N') + P_1(N'') \)
  most profitable split for total productivity: \( N' \approx N'' \)

  \( \Rightarrow \) critical ("split") size: \( N_{\text{crit}} = 2(g_1 + c_1)/3c_1 = 4/3 \; N_{\text{opt}} \)
so far: optimization of *global* productivity

- realistic system: cannot probe all possible configurations
  - no breaking/rearrangement of large number of connections
- growing systems: more likely follow established pathways
  - search for “local” optima (rather than global optima)
  - different set of growth rules lead to high productivity

**Example 1:**

1. simultaneous formation of isolated groups until $P_1/N$ decreases
2. groups interact → formation of supergroups until $P_2/N$ decreases
3. supergroups interact → formation of super-supergroups until $P_3/N$ decreases
4. groups at level $k - 1$ can still grow further if this increases productivity at level $k$
Formation of Hierarchies

(Drossel '99)
Example 2:

1. level 1: add new units → (i) join existing groups if this increases productivity, OR (ii) form a new group with one of the units in other groups, as long as productivity increases

2. migration of units from other groups to newly formed one, as long as productivity increases

3. level $k$: split of groups from supergroups to form new supergroups, and migration of groups to other supergroups, as long as productivity increases
Formation of Hierarchies

N=5, P/N=2.0
\[
\begin{array}{c}
5 \\
\end{array}
\]

N=6, P/N=2.68
\[
\begin{array}{c}
3 \\
\end{array}
\]

N=14, P/N=7.94
\[
\begin{array}{c}
4 \\
5 \\
\end{array}
\]

N=41, P/N=20.8
\[
\begin{array}{c}
7 \\
7 \\
6 \\
\end{array}
\]

N=42, P/N=21
\[
\begin{array}{c}
7 \\
7 \\
7 \\
\end{array}
\]

N=43, P/N=25.4
\[
\begin{array}{c}
6 \\
6 \\
6 \\
\end{array}
\]

N=146, P/N=116
\[
\begin{array}{c}
8 \\
8 \\
8 \\
8 \\
7 \\
7 \\
8 \\
\end{array}
\]

N=168, P/N=60.3
\[
\begin{array}{c}
9 \\
9 \\
9 \\
9 \\
9 \\
9 \\
9 \\
\end{array}
\]

(Drossel '99)
Example 3: (Drossel '99)

1. add new units to a group until productivity decreases
2. split into two groups that grow until productivity decreases
3. rearrangement into three groups that grow until productivity decrease
4. split into two supergroups that grow until productivity decreases
Formation of Hierarchies

(Drossel '99)
Result:

- complexity emerges: formation of hierarchies to optimize two contradicting requirements (benefit vs. costs of interaction)

Extensions:

- heterogeneous agents: no identical units, groups, ....
- explicite time dependence: “aging of groups”
- explicite dependence on distance, costs of migration
- dynamics of entry/exit: “birth” dependent on local conditions, “death” of units, groups, ...
- dependence on resources
Some References

References