Multiplicative Models for Company Dynamics

Frank Schweitzer
fschweitzer@ethz.ch

Thanks to …
S. Jain, L. Amaral, H. Takayasu, A. Seufert, N. Bürkler...
Dynamics of Companies: Firm Model Tree of its Research Evolution
Company Growth

- set of companies: $i = 1, \ldots, N$
  - $x_i(t)$: company “size” (income, output, employees, ...)
  - growth rate: $dx_i/dt = F_i(t)$
- $F_i(t)$ with $\langle F_i(t) \rangle = 0$, $\langle F_i(t)F_i(t') \rangle = S \delta_{ij} \delta(t - t')$
  
  $$x_i(t + \Delta t) = x_i(t) + \sqrt{S \Delta t \xi_i}$$

- growth as random walk (Bachelier, 1900)

- independent growth, proportional to size (Gibrat, 1931)

- growth through innovation networks
Growth through Network Effects

\[ \dot{x}_i = \mathcal{F}_i = f(x_j, x_k) + \ldots \]

\[ \frac{dx_i}{dt} = \sum_{j=1}^{N} c_{ij} x_j - \Phi x_i \]  

(Jain/Krishna '98, '01)

- \( c_{ij} \in \{0, 1\} \Rightarrow \) represents a directed network
  - \( j \) catalyzes the growth of \( i \), link probability \( p \)
  - \( i \) is connected to \( m = p(N - 1) \) other companies (on average)

- **two time scales:** company growth (fast), network dynamics (slow)
- **assumption:** extremal dynamics \( \Rightarrow \) minimum performance threshold

- **questions:**
  - Under which conditions do companies survive?
  - Which structures of innovation networks emerge?
  - What happens if selection pressure is increased?
**Result:** Emergence of a core of *cooperative* companies, and a *parasitic* periphery, considerable crashes and recovery
**Multiplicative Growth**

- \[ \dot{x}_i = \mathcal{F}_i = f(x_i) + \ldots = b x_i \]
  - “Law of proportionate growth” (Gibrat ’30, ’31; Sutton ’97)
  - no interactions between firms

\[ x_i(t + \Delta t) = x_i(t) \left[ 1 + b_i(t) \right] \]

- Assumptions:
  - \( b_i(t) \): independent of \( i \), no temporal correlations (random noise)
  - *growth “rates”*: \( R(t) = x(t + 1)/x(t) \), \( t \gg \Delta t \), \( \ln(1 + b) \approx b \)

\[ \ln R(t) = \sum_{n=1}^{t} b(n) \]

\[ \Rightarrow \text{random walk for } \ln R(t) \Rightarrow \text{log-normal distribution for } x_i(t) \]

- empirical evidence?

Fig. 2(a)
Empirical distribution of growth rates (Amaral et al, 1997) ⇒ depend on size, tent-shape exponential distribution
Empirical distribution of standard deviation of growth rates (Amaral et al, 1997) depend on size, power-law distribution
Stylized facts:

➤ log-normal distribution of company sizes

\[ P(x) = \frac{1}{\sqrt{2\pi} \sigma x} \exp \left[ \frac{-(\ln x - \mu)^2}{2\sigma^2} \right] \]

➤ exponential growth ratio distribution

\[ P(r_1|x_0) = \frac{1}{\sqrt{2} \sigma_1(x_0)} \exp - \left[ \frac{\sqrt{2} |r_1 - \bar{r}_1(x_0)|}{\sigma_1(x_0)} \right] \]

➤ power-law distribution of the standard deviations

\[ \sigma_1(x_0) \sim x_0^{-\beta} \; ; \; \beta < 0.5 \]
Explanations:

➤ correlations in the growth rates

company is attracted to an “optimal size”

\[
\frac{x_{t+\Delta t}}{x_t} = \begin{cases} 
  ke^{e_t}, & x_t < x^* \\
  \frac{1}{k}e^{e_t}, & x_t > x^*,
\end{cases}
\]

➤ growth depends on properties of management hierarchies

\(n\) levels, \(z\) units, decisions on higher level are followed with prob \(\pi\)

\[
\beta = \begin{cases} 
  -\ln(\pi)/\ln(z) & \text{if } \pi > z^{-1/2} \\
  1/2 & \text{if } \pi < z^{-1/2}
\end{cases}
\]

- \(\beta\) decreases in time \(\Leftrightarrow\) companies better coordinated
Further Improvements of Gibrat’s Model

➤ **economic idea:** simple entry dynamics (Simon & Bonini ’58)

➤ **mathematic idea:** *add more noise!* (Kesten ’73)

\[ x(t + 1) = x(t)[1 + b(t)] + a(t) \]

- \( b, a \) positive independent random variables
- \( a(t) \) acts as “effective repulsion” from zero (Sornette & Cont ’97)

➤ **practical idea:** fit parameters (Takayasu et al ’03)

\[ x(t + 1) = \alpha(t)\lambda(t, x) x(t) + a(t) \]

- \([1 + b] \to \lambda(x, t)\): growth depends on size
- estimation from \( \ln R(t) = \ln\{x(t + 1)/x(t)\} \) with standard deviation \( \sigma(x) \)
Comparison with real company data

➤ Takayasu et al ‘03: income of 15,000 US and 15,000 non-US comp., 80,000 Japanese comp. (income > 40 Mio Yen), before tax
**Estimation of** ln $R(t)$, $\sigma(x)$

- for large $x$: $\sigma_0$, $f(t)/x$ negligible
- scaling by means of normalized growth: $R^{\sigma(x)/\sigma_0}$

(Takayasu et al '03)


\( \alpha(t) \) either +1 (growth) or (-1) (slump)

prob. determined empirically from large \( |x(t)| \)

\[
\alpha(t) = \begin{cases} 
1 & \text{with prob. 0.97 (} x(t) > 0), \ 0.75 \ (x(t) < 0) \\
-1 & \text{with prob. 0.03 (} x(t) > 0), \ 0.25 \ (x(t) < 0) 
\end{cases}
\]
Forecast by means of Monte Carlo Simulations

- initial state: 6,000 companies, $x_i(0) = 100$
coefficients estimated from real data

- $t = 50$: qualitative agreement with real distribution (US)
with constant growth rate distribution: firms income will keep growing
for more than 100 years
Investment Strategies?

- normalized cumulative income for 5 years: \( I = \sum_{n=1}^{5} \frac{x(n)}{x(0)} \)

- for \( x(0) > 10^6 \text{\$} \): \( I \propto x(0) \Rightarrow \text{invest in small firms?} \)
**Topics for Future Investigations**

1. Growth strategies

➤ companies with growth strategies: \( b(t) \rightarrow r(t)q_i(t) \)

\[
x_i(t + \Delta t) = x_i(t) \left[ 1 + r(t)q_i(t) \right] + a
\]

➤ \( r(t) < 0 \Rightarrow q_i(t) \rightarrow q_{\text{min}}, \quad r(t) > 0 \Rightarrow q_i(t) \rightarrow q_{\text{max}} \)

- Can company predict \( r(t) \)? Can it adjust \( q_i(t) \) fast enough?
2. Stochastic Models with interactions

➤ combining multiplicative growth models and network models

\[ x_i(t + 1) = x_i(t) \left[ 1 + b_i(t) + \sum_{j=1}^{N} c_{ij} x_j \right] + G(t) - L(t) \]

- network dynamics: \( c_{ij} \Rightarrow c_{ij}(t, r) \)
  represents spatial interaction, cooperation, spread of innovations, ...
- “catalytic” growth: \( b_i(t) = r(t) q_i(t) \)
  represents strategic decisions, organizational efficiency, ...
- \( G(t), L(t) \) represent external conditions (gov, environment, ...),
global couplings (limited ressources, ...)
Conclusions

➤ *company growth*: field of applications for stochastic processes
  • network effects? ⇒ innovation economics, input from physics?

➤ *multiplicative growth models*: are successfully applied to real data
  • stylized facts reproduced
  • conceptual drawbacks: random growth ⇔ firms as profit maximizers respond similarly to changing market conditions

➤ *micro models* ??
  • hierarchical-tree model (Amaral et al, 1997)
  • multi-agent models (Axtell, ...)
  • heterogeneity? market ecology of different organizational forms