



Recombinant knowledge and the evolution of innovation networks

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ABSTRACT

We introduce a new model for the evolution of networks of firms exchanging knowledge in R&D partnerships. Innovation is assumed to result from the recombination of knowledge among firms in an R&D intensive industry. The decision of two firms to establish a new partnership or to terminate an existing one, is based on their marginal revenues and costs, which in turn depend on the position they occupy in the network. Moreover, the formation of a collaboration has significant external effects on the other firms in the same connected component of the network. We show that this decentralized partner selection process leads to the existence of multiple equilibrium structures. Finally, by means of computer simulations, we study the properties of the emerging equilibrium networks and we show that they reproduce the stylized facts of R&D networks.

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1. Introduction

R&D partnerships have become a widespread phenomenon characterizing technological dynamics, especially in industries (Hagedoorn, 2002; Letterie et al., 2008) with rapid technological development such as, for instance, the biotechnology, chemical and computer industries (see Ahuja, 2000; Pammolli and Riccaboni, 2002; Powell et al., 2005; Roijakkers and Hagedoorn, 2006). These industries have been characterized by a process of convergence of firms' technological competencies (see e.g. Gambardella and Torrisi, 1998; Nesta and Dibiaggio, 2003; Patel and Pavitt, 1997). At the same time, in these industries, the knowledge base involves general and abstract knowledge often built on scientific research (e.g. immunology and molecular biology in the biotechnology sector, see Powell et al., 1996). This has allowed for a division of innovative labor and fostered collaboration across firms (see Arora and Gambardella, 1994a, 1994b). In this regard, firms in these sectors display significant heterogeneity in the way they combine their homogeneous technological competencies to develop new products and processes or new technological fields (see e.g. Antonelli et al., 2010; Nesta and Dibiaggio, 2003).

In this paper, we present a model in which innovation stems from the heterogeneous recombination of firms' technological competencies, via a network of costly R&D collaborations. Within this framework, we study the emergence of pairwise stable structures by employing the notion of "improving path" (cf. Jackson and Watts, 2002), and assuming that link deletion is subject to severance costs. In particular, we show the existence of multiple stable structures for the same level of collaboration cost. In addition, we investigate equilibrium selection under a two-sided myopic link dynamics

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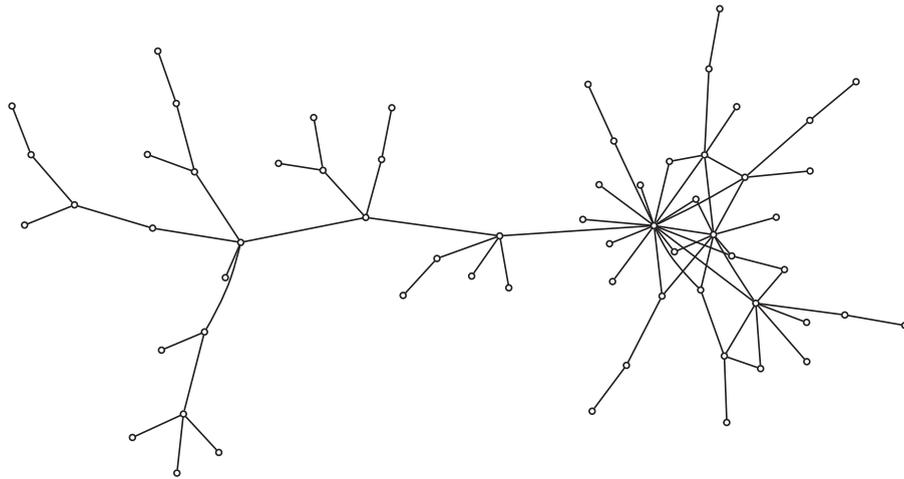


Fig. 1. Example of an empirical R&D network. A connected component of R&D alliances in the biotech sector established in the year 2000. The alliance information is taken from the Thomson SDC Platinum database.

and we show that the model is able to generate stable structures that match the properties of empirically observed R&D networks.

Our research is motivated by a recent stream of literature which has investigated the salient features of empirically observed R&D networks (see e.g. Ahuja, 2000; Fleming et al., 2007; Hanaki et al., 2010; Powell et al., 2005; Roijakkers and Hagedoorn, 2006). These empirical studies have identified three main structural properties of innovation networks that are invariant across the different industries examined: (i) networks are sparse, that is, from all possible connections between firms, only a small subset is realized. (ii) Networks are highly clustered, that is, they are locally dense. In clusters, firms are closely interconnected but between different clusters there exist only few connections. (iii) The distribution of links over the firms tends to be highly heterogeneous with only few firms being connected to many others. A typical example of an R&D network featuring these characteristics can be seen in Fig. 1. Following this wave of empirical research, theoretical models have explored the emergence of R&D networks in a framework with firms being allowed to form any arbitrary pattern of bilateral R&D agreements (Cowan et al., 2006; Goyal and Joshi, 2003; Goyal and Moraga-Gonzalez, 2001). However, these models lead to network structures that are too simple to account for the stylized facts listed above.

In this paper, we propose a new model of the evolution of R&D networks which contributes to the foregoing literature along several lines. First, the main driving factor of the growth of knowledge is the recombination of existing knowledge among collaborating firms (see Ahuja, 2000; Kogut and Zander, 1992; Meagher and Rogers, 2004; Powell et al., 1996; Weitzman, 1998). We show that this implies that not only the connections to the first neighbors or the shortest paths between firms play a role in the recombination of knowledge, but also instead all paths are important. Second, we show that multiple equilibrium structures can coexist for the same level of collaboration cost. To this end, we demonstrate that, both the spanning star (i.e. the star encompassing all nodes in the network), and the graph composed of disconnected cliques of the same size are possible equilibrium networks. Moreover, we investigate systematically the equilibria emerging under a two-sided myopic link dynamics (Jackson and Watts, 2002) and we show that the model is able to generate stable structures that match the properties of empirically observed R&D networks. In particular, the equilibrium networks we find are sparse, clustered and characterized by large variance in the degree distribution. Interestingly, the last feature is obtained despite the fact that the model does not assume a preferential attachment mechanism.

The idea that innovations stem from the possibility of recombining firms' knowledge stocks is formalized by assuming that (i) the number of innovations is proportional to knowledge stock of the firm (see e.g. Cincera, 1997; Hall et al., 1986; Pakes and Griliches, 1984), and that (ii) firm's knowledge growth is an increasing function of the individual knowledge stock of the firm and the knowledge of its R&D partners (Cohen and Levinthal, 1989; Dosi, 1982, 1988).

On the other hand, firm's discounted profits are a function of the number of innovations, given the current structure of the network, and include the costs of R&D collaborations. Unit cost of collaborations is homogeneous across firms and indirectly account for the average strength of barriers to technological transfers across firms. Interestingly, as result of hypotheses above, revenues from innovations are proportional to the largest eigenvalue of the adjacency matrix associated with the connected component to which the firm belongs (see Proposition 1 and Eq. (9)). The largest eigenvalue is related to the number of all walks connecting firms in a given component (Cvetkovic et al., 1997). This has several implications. As the largest eigenvalue is the same for all firms in the same component, the formation/deletion of a collaboration by a firm has a strong non-rival external effect on all its direct and indirect neighbors. However, the deci-

sion whether a link is formed or deleted depends on the change in eigenvalue which instead varies with the position of the two firms involved in the collaboration. This implies a strong path-dependent character of partner's choice decisions.

Our analysis of R&D network dynamics differs from previous ones in the literature in another two important respects. First, motivated by the evidence on the increasing technological convergence in R&D collaboration intensive industries, we do not analyze the technological differences between pairs of partners in the formation/destruction of alliances between two partners (e.g. Carayol and Roux, 2009; Cowan and Jonard, 2007). However, by varying the degree of collaboration cost in the industry, we study how the network of collaborations evolve when knowledge exchange is on average easier or harder in the industry. Another specificity of our model is that in the evolution of the network, which is based on the notion of "improving path" (cf. Jackson and Watts, 2002), we introduce an asymmetry between link creation and link deletion. In particular, there is a severance cost involved in link deletion, which turns out to be important for the emergence of non-trivial network structures.

Our model can also be related to models of the strategic network formation literature in which agents face a trade-off between the benefit they get from accessing the network and the cost of forming links with other agents (see e.g. Bala and Goyal, 2000; Carayol and Roux, 2009; Carayol et al., 2008; Haller and Sarangi, 2005; Jackson and Wolinsky, 1996; Vega-Redondo and Goyal, 2007). To this regard, our model shares many similarities and differences with the "connections" model introduced in Jackson and Wolinsky (1996) and with the linear "two-way flow" model without decay introduced in Bala and Goyal (2000). For instance, similar to both models, the benefit an agent receives from the network derives also from indirect connections. However, an important difference with respect to both models is that here all walks in the graph matter. This implies that we obtain different stable structures with respect to those models. In particular, we find that, for high level of cost, symmetric but disconnected structures are stable, in contrast with Jackson and Wolinsky (1996) who find asymmetric but connected structures (the star) to be stable. In addition, the benefit of a collaboration is non-rival (see in particular Eq. (9) and the discussion thereafter).¹

The paper is organized as follows. Section 2 contains the description of the model, starting with the definition of the network of R&D collaborations across firms, and then moving to explain how firms profit from R&D collaborations. In Section 3 the network formation, the emergence of equilibrium networks and their properties are analyzed. Finally, Section 4 concludes. All proofs can be found in Appendix A.

2. The innovation process

We consider an industry in which firms engage in pairwise R&D collaborations with other firms. Collaborations allow the growth of knowledge within the firm and an increase in the number of innovations that yield profits to the firm. We first define the network of R&D collaborations. Next, we characterize how the R&D network influences knowledge growth, innovation and profits of the firms.

2.1. The network

Consider an industry populated by n firms. The *network*² G is the pair (N, E) consisting of the set of nodes $N = \{1, \dots, n\}$, representing the population of firms and a set of edges E , representing R&D collaborations among the firms. We consider undirected graphs only (for simplicity we may just write N and E where it is obvious to which network G the sets refer). Undirected graphs map the hypothesis that both partners engaged in a collaboration exchange knowledge and improve their knowledge stocks. An edge $ij \in E$, represents the existence of an R&D collaboration between firms i and j , which are said to be *adjacent*. A *subgraph* of G is a pair $G' = (N', E')$ such that $N' \subseteq N$, $E' \subseteq E$. The number of nodes is $|N'| = n$ and the number of edges $|E'| = m$. A *complete graph* K_n is a graph in which all n nodes are pairwise adjacent. The graph in which no pair of nodes is adjacent is the empty graph \bar{K}_n . A *clique* $K_{n'}$, $n' \leq n$, is a complete subgraph of the network G . In contrast to the clique, an *independent set* $\bar{K}_{n'}$ is a subgraph in which all n' nodes are not pairwise adjacent. The *neighborhood* of node i is the set $N_i = \{j \in N : ij \in E\}$. The *degree* of a node i in G , written by d_i , is the number of edges incident to i . Clearly, $d_i = |N_i|$. The maximum degree is $\Delta(G)$ and the minimum degree is $\delta(G)$. The *clustering coefficient* C_i of firm i is the proportion of links among the firms within its neighborhood N_i divided by the number of links that could possibly exist between them, i.e.

$$C_i = \frac{|\{jk : j, k \in N_i \wedge jk \in E\}|}{d_i(d_i - 1)/2}. \quad (1)$$

The total clustering coefficient is the sum of the clustering coefficients for each firm, $C = \sum_{i=1}^n C_i$. A *walk* W_k of length k connecting firm i_1 and i_k is a sequence of firms (i_1, i_2, \dots, i_k) such that $i_1 i_2, i_2 i_3, \dots, i_{k-1} i_k \in E$. A walk is *closed* if the first and last firm in the sequence are the same, and *open* if they are different. A *path* is a walk in which no firm is visited twice. A closed

¹ See Vega-Redondo and Goyal (2007) for a model where benefits from indirect connections are rival.

² In this paper we will use the terms graph and network interchangeably. The same holds for links and edges.

path encompassing n nodes is a cycle, denoted by C_n . A *connected component* in G is a maximal set of firms such that there exists a path between any two of them. We will say that two components are *disconnected* if there is no path between them. A *connected graph* is a graph consisting of only one connected component. A graph is *minimally connected* if the removal of any link makes it disconnected. Let $A(G)$ be the symmetric $n \times n$ *adjacency matrix* of the R&D network G . The element $a_{ij} \in \{0, 1\}$ indicates if there exists a link between firm i and j such that $a_{ij} = 1$ if $ij \in E$ and $a_{ij} = 0$ if $ij \notin E$. The *eigenvalues* of the adjacency matrix \mathbf{A} are the numbers λ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ has a nonzero solution vector, which is an *eigenvector* associated with λ . The term λ_{PF} denotes the *largest real eigenvalue* of \mathbf{A} (the *Perron–Frobenius eigenvalue*, cf. Horn and Johnson, 1990; Seneta, 2006), i.e. all eigenvalues λ of $\mathbf{A}(G)$ satisfy $|\lambda| \leq \lambda_{PF}$ and there exists an associated nonnegative eigenvector $\mathbf{v} \geq 0$ such that $\mathbf{A}\mathbf{v} = \lambda_{PF}\mathbf{v}$. For a connected graph G the adjacency matrix $A(G)$ has a unique largest real eigenvalue λ_{PF} and a positive

associated eigenvector $\mathbf{v} > 0$. Finally, for a graph with n nodes there are $\binom{n}{2}$ possible links and accordingly there are $2^{\binom{n}{2}}$ possible graphs on n nodes. We denote with $G(n, p)$ the *random graph* with n nodes, in which each of the possible links occurs independently with probability p . Similarly, the random graph $G(n, m)$ with n nodes and m edges can be defined (Bollobas, 1985).

2.2. Innovation and profits from R&D collaborations

Firms exploit R&D collaborations to introduce innovations in the industry. New knowledge is generated over time by recombining the existing knowledge stocks of firms in the economy via the existing network of R&D collaborations (see Kogut and Zander, 1992; Meagher and Rogers, 2004; Weitzman, 1998). Let us denote by $x_i(t)$ the stock of knowledge of firm $i \in N$ at time $t \geq 0$ and let $A(G(t))$ be the adjacency matrix with elements $a_{ij}(t)$, corresponding to the network $G(t) = (N, E(t))$ of R&D collaborations. Then new knowledge within firm i is generated according to

$$\dot{x}_i(t) = \gamma x_i(t) + \sum_{j=1}^n a_{ij}(t) x_j(t), \quad i = 1, \dots, n. \quad (2)$$

The factor $\gamma > 0$ takes into account the influence of firm i 's own stock of knowledge x_i on its knowledge growth. Thus, we assume $a_{ii}(t) = 0$ for all i and t . In vector-matrix notation Eq. (2) reads $\dot{\mathbf{X}}(t) = (\mathbf{A}(G(t)) + \gamma \mathbf{I})\mathbf{X}(t)$. Note also that in Eq. (2), for non-negative initial values of knowledge stocks, $\mathbf{x}(0) \geq 0$, we have that $\dot{\mathbf{X}}(t) \geq 0$ as well as $\mathbf{X}(t) \geq 0$.

The growth of knowledge of firm i in Eq. (2) is directly affected by the number of neighbors i has, as well as by the stock of knowledge of its neighbors. Therefore, the topology of the whole network of R&D collaborations (including all direct and indirect paths along which knowledge can flow between firms) influences the innovation process within the firm. The idea that knowledge growth is positively related to knowledge stocks (of the firm and of its R&D partners) captures the property of *cumulativeness* in technical change processes (see Dosi, 1982, 1988; Freeman, 1994). Furthermore, the fact that firms rely both on internal and on external sources for knowledge growth is a robust stylized fact in the economics of technical change (see e.g. Arora et al., 2004; Cohen and Levinthal, 1989; Freeman, 1994). Finally, many empirical studies about R&D networks (e.g. Ahuja, 2000; Powell et al., 1996) indicate that the rationale for R&D collaborations lies more in the possibility of having access to knowledge spillovers, rather than in the strategic choice of sharing the cost of R&D investments across different organizations.³

The next proposition establishes a relation between, on the one hand, the asymptotic growth rate of knowledge, the asymptotic relative stock of knowledge and the rate of convergence, and, on the other hand, the eigenvalues and eigenvectors of the adjacency matrix $\mathbf{A}(G_i(t))$ of the connected component $G_i(t)$ of firm i in the network $G(t)$.

Proposition 1. Assume that $G(t + \tau) = G(t)$ for any $\tau \geq 0$. Consider the eigenvalues $\lambda_{PF} = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ associated with the adjacency matrix $\mathbf{A}(G_i(t))$ of the connected component $G_i(t) \subseteq G(t)$ of firm $i \in N$. Then the following properties hold:

- (i) The asymptotic knowledge growth rate of a firm i is constant and equal to the largest real eigenvalue (Perron–Frobenius eigenvalue) of the adjacency matrix $\mathbf{A}(G_i(t))$:

$$\lim_{\tau \rightarrow \infty} \frac{\dot{x}_i(\tau)}{x_i(\tau)} = \lambda_{PF}(G_i(t)) + \gamma, \quad i = 1, \dots, n. \quad (3)$$

The rate of convergence is $\mathcal{O}(e^{-[\lambda_{PF}(G_i(t)) - \lambda_2(G_i(t))]\tau})$ as $\tau \rightarrow \infty$.

³ All the above mentioned studies do not specify a precise functional relation between on the one hand knowledge growth and internal and external knowledge stocks on the other hand. Nevertheless, the linear functional form in Eq. (2) is both simple and allows for analytical tractability of the model. In addition, knowledge-growth functions similar to Eq. (2) can be found in previous theoretical R&D related studies (e.g. Padgett et al., 2003; Spence, 1984).

(ii) The asymptotic value of a firm i 's relative knowledge stock equals the element v_i of the eigenvector associated with the eigenvalue $\lambda_{PF}(G_i(t))$:

$$\lim_{\tau \rightarrow \infty} \frac{x_i(\tau)}{\sum_{j=1}^n x_j(\tau)} = v_i, \quad i = 1, \dots, n. \tag{4}$$

Part (i) of the above proposition states that the knowledge dynamics defined in Eq. (2) converges, for a given R&D network, to a steady state characterized by a constant growth rate of knowledge. In addition, such a constant growth rate depends on the topology of the connected component which the firm belongs to (through the largest eigenvalue $\lambda_{PF}(G_i(t))$). Moreover, part (ii) implies that the topology of the connected component $G_i(t)$ determines the distribution of relative values of the knowledge stocks of firms in the same component. Finally, the rate of convergence to the steady state is determined by the eigenvalues of $A(G_i)$.⁴

Similarly to Goyal and Moraga-Gonzalez (2001) and Goyal and Joshi (2003) we assume that firms use their accumulated knowledge to introduce innovations in the industry. More precisely, grounded on a significant body of empirical research (see e.g. Cincera, 1997; Hall et al., 1986; Pakes and Griliches, 1984) we assume that returns $R(x_i(t))$ from innovations at time t are an increasing and concave function of the knowledge stock $x_i(t)$ of firm i according to⁵:

$$R(x_i(t)) = V \ln x_i(t), \quad V > 0. \tag{5}$$

Each collaboration involves an increasing cost per unit of time $\tilde{c}t$.⁶ If firm i engages in $d_i(t)$ collaborations at time t then its total costs per unit of time is equal to $d_i(t)\tilde{c}t$. Higher or lower values of unitary costs \tilde{c} map situations in which knowledge is, on average, more or less difficult to transfer across firms in the industry. In the next sections we study the effect on the network dynamics for different levels of these unitary collaboration costs.

When forming or severing a collaboration firms evaluate the future discounted profit they will gain, assuming that the new network will not change over time. From now on, let t denote the last instant in which the network has changed and τ the time variable after that change. A firm assumes that $G(t) = G(t + \tau)$ for any $\tau \geq 0$. It then follows that total discounted profits from innovation of firm i from collaborations with $d_i(t)$ other firms are equal to

$$\tilde{\pi}_i(G_i(t), \tilde{c}, t) = \int_0^\infty (V \ln x_i(t + \tau) - \tilde{c}d_i(t)(t + \tau))e^{-\rho\tau} d\tau, \tag{6}$$

where $\rho > 0$ is the discount rate. Given the results in Proposition 1, we can now state the following proposition, linking discounted returns from innovations to the eigenvalue of the adjacency matrix $A(G_i(t))$.

Proposition 2. *Let $G_i(t)$ be the connected component of firm i in the network $G(t)$. Then firm i 's discounted revenue from innovations is given by*

$$V \int_0^\infty \ln x_i(\tau) e^{-\rho\tau} d\tau = \frac{V(\lambda_{PF}(G_i(t)) + \gamma)}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho}\right). \tag{7}$$

Proposition 2 states that if firms are weakly discounting future revenues (i.e. as $\rho \rightarrow 0$) then their present value of future revenues is, at first order in $1/\rho$, determined by the largest real eigenvalue of their connected component.⁷

In light of the above considerations, we assume that ρ is small. In this case we can approximate total discounted profits with the following expression:

$$\tilde{\pi}_i(G_i(t), \tilde{c}, t) \approx \frac{V(\lambda_{PF}(G_i(t)) + \gamma)}{\rho^2} - \frac{\tilde{c}d_i(t)}{\rho^2}. \tag{8}$$

Since the discount rate is identical for all firms, we apply an affine transformation to obtain the final expression for the payoff of firm i from R&D collaborations

$$\pi_i(G_i(t), c, t) = \lambda_{PF}(G_i) - cd_i(t), \tag{9}$$

where $c = \tilde{c}\rho^2/V$. Eq. (9) implies that the payoff from collaborations depends on the topology of the network. In particular, the largest eigenvalue associated with a component, $\lambda_{PF}(G_i)$, coincides with the growth rate in the number of walks of length k in a component, when the length is increased by one (see Cvetkovic et al., 1997, p. 24). Interestingly, empirical studies on

⁴ In general, the convergence in a connected component to its largest real eigenvalue is always guaranteed. In addition, more dense networks are characterized by a faster convergence (see Proof of Proposition 1 in Appendix A). However, the convergence can be slow for sparse networks and particular network topologies (see also Jackson and Golub, 2007).

⁵ The logarithmic form of the return function in Eq. (5) is both increasing and concave in the knowledge level. In addition, it allows in a simple way the convergence of discounted revenues from innovation in Eq. (6).

⁶ The assumption that costs increase over time during a collaboration, avoids that total discounted costs become negligible compared to revenues in Eq. (6). In addition, the fact that R&D costs display a positive trend is also documented by empirical evidence (see e.g. Copeland and Fixler, 2009; DiMasi et al., 2003; Jones, 1995).

⁷ The approximation in Eq. (7) has also been tested numerically for several classes of networks, including those analyzed in the paper. These tests show that the approximation of firm's present value of revenues with the largest eigenvalue is very good for values of $\rho \in [0.01, 0.2]$.

R&D network formation support the idea that firms establish R&D collaborations in a way that increases the number of all walks in the network (cf. “multiconnectivity” in Powell et al., 2005). On the other hand, the payoff decreases with the degree d_i of the firm. Therefore, it is best for a firm to reach the other firms through many walks but to have not too many links to pay for.

The following lemma characterizes the relation between the largest eigenvalue of a connected component and the creation or removal of R&D collaborations.⁸

Lemma 1. Denote $G' = (N', E')$ the graph obtained from the graph $G = (N, E)$ by the addition or removal of an edge ij . Then

- (i) $\lambda_{\text{PF}}(G') \geq \lambda_{\text{PF}}(G)$ if $ij \notin E$ and $\lambda_{\text{PF}}(G') \leq \lambda_{\text{PF}}(G)$ if $ij \in E$.
- (ii) $\lambda_{\text{PF}}(G) \leq \lambda_{\text{PF}}(K_n) = n - 1$.
- (iii) $\lambda_{\text{PF}}(G') - \lambda_{\text{PF}}(G) \leq 1$.

Thus, the largest real eigenvalue in a component is non-decreasing in the number of links. In addition, it is bounded, since its value can never be higher than the one associated with the complete graph K_n . Finally, the change in the eigenvalue is itself bounded, since its value must be less than one. The preceding observations deliver two central properties of the model. First, since the long run knowledge growth rate and net present revenues from innovations are the same for all the firms in a given connected component, the creation (deletion) of a collaboration by one firm has a positive (negative) non-rival external effect on all its direct and indirect neighbors in the component. Second, the marginal revenue from R&D collaborations is always an increasing (albeit bounded) function of the number of links. This means that the creation (deletion) of a new R&D collaboration increases (decreases) the number of innovations and thus the discounted revenue. Moreover, the revenue itself is a bounded function of the number of links. The last property does not imply that the revenue is also a concave function of the number of links.⁹ However, it implies that, as the network grows in the number of links, the highest marginal revenue that can be obtained from the creation of a new link or from the removal of an existing link can become very small. It turns out that, when the highest marginal revenue from a collaboration is smaller than the marginal cost of collaboration, the network reaches an equilibrium, and this may happen well before the network has grown to a fully connected graph.

3. Network evolution

In this section we investigate how the network structure evolves when firms are allowed to endogenously choose their partners on the basis of marginal profits presented in the previous section. Following Jackson and Wolinsky (1996), we consider a network formation process in which the creation of a new link requires the bilateral agreement of the two parties involved. By contrast, the deletion of a link only requires the unilateral decision of one of the two firms. Consistent with this link creation and deletion mechanism, we adopt the definition of pairwise stability as a network equilibrium criterion (Jackson and Wolinsky, 1996). Based on this definition of stability, we first derive the conditions on the value of cost for which certain network structures are stable. Among the possible stable graphs, we find a disconnected graph consisting of multiple cliques of the same size, as well as a spanning star. Moreover, we show that these structures can be both stable for the same values of cost, thus, implying a co-existence of multiple equilibrium networks.

The above mentioned relatively simple structures are not the only stable networks emerging in our model. Since it is increasingly difficult to derive general proofs of stability for more complex networks, we perform a dynamic study of network stability.¹⁰ We model explicitly the evolution process in which, at the beginning of each period, a pair of firms decides whether to form or delete a link, based on the discounted marginal profits this action brings about. This investigation, performed through computer simulations, shows that there exist a multitude of complex structures which are pairwise stable. Remarkably, these networks display topological properties that are consistent with the stylized facts of R&D networks in a region of the parameters of the model.

3.1. Improving paths and equilibrium networks

We consider a process of network evolution in which firms form or delete one link at a time based on the marginal profits they expect from that action. In other words, new links are created whenever the increase in marginal revenue of a new collaboration is greater than the marginal cost of a collaboration, with the gain being strict for at least one of the firms in the selected pair. Likewise, link deletion occurs whenever the savings in marginal cost from removing a collaboration are enough to compensate for the decrease in marginal revenue. However, given its unilateral nature, we assume that removing a collaboration involves severance costs so that the savings in marginal costs from removing a collaboration are reduced by a factor α . Severance costs are the extra-costs that a firm faces when unilaterally terminating an R&D collaboration. They can

⁸ A proof of the foregoing lemma can be found in Cvetkovic et al. (1995).

⁹ Incidentally, note that λ_{PF} is not even determined as a function of m , because, for a given m , there are many different ways to arrange the links among the nodes, resulting in different values of λ_{PF} .

¹⁰ See the discussion in Vega-Redondo (cf. 2007, p. 208).

arise from different sources. For instance, [Saxenian \(1994, p. 164\)](#) describes how in Silicon Valley firms sign cross-licensing agreements to avoid costly patent litigations. Breaking up these agreements can expose firms to costly lawsuits by their previous partners. Likewise, [Kogut \(1989\)](#) shows that the likelihood of termination of a joint venture is significantly reduced if the joint venture is bundled with a series of other agreements (e.g. licensing, supply contracts) involving the same partners. This is because non-cooperative behavior by a joint venture partner is punishable by reciprocating in the context of the other agreements. Furthermore, many empirical works on R&D collaborations (see e.g. [Gulati, 1995](#); [Gulati and Sytch, 2008](#); [Ring and Van de Ven, 1992](#), among many others) have shown that the existence of a previous network of collaborations plays a key role in the governance of a collaboration. In particular, networks form a basis for the emergence of mechanisms of “deterrence-based” trust (see [Shapiro et al., 1992](#)). Namely, the fear of losing future business interactions with the ongoing partner or its close neighbors, or the fear of losing reputation of behaving cooperatively, is higher if the R&D collaboration is embedded in a network of other collaborations. In turn, this discourages opportunistic behavior within a collaboration and creates “self-enforcing” safeguards that can substitute for contractual safeguards ([Gulati, 1998](#); [Powell, 1990](#)).

Following [Jackson and Watts \(2002\)](#), we call *improving path*, a sequence of networks $(G(t))_{t \in \mathbb{N}}$ such that (i) any two consecutive networks, $G(t)$ and $G(t+1)$, differ by one link only, (ii) if the link is added, both firms benefit from the new link, at least one of them strictly, and (iii) if a link is deleted, at least one of the two firms strictly benefits from the deletion.

Let G denote the current graph at time t . Further, denote by $G+ij$ the graph obtained from G by adding the edge ij . Similarly, let $G-ij$ denote the graph obtained by removing the edge ij . Denote by $\lambda_i(G)$ the largest eigenvalue $\lambda_{\text{PF}}(G_i)$ of the connected component G_i to which the firm i belongs. Note that, although link deletion implies that the degree of firm i is reduced by one, the firm saves only a fraction of the cost due to the presence of the severance costs $\nu(c) = (1-\alpha)c$. Thus, the change in profits of firm i induced by the removal of a link is given by

$$\begin{aligned} \pi_i(G-ij) - \pi_i(G) &= \lambda_i(G-ij) - (d_i - 1)c - \nu(c) - (\lambda_i(G) - d_i c), \\ &= c - \nu(c) - (\lambda_i(G) - \lambda_i(G-ij)), \\ &= \alpha c - (\lambda_i(G) - \lambda_i(G-ij)), \end{aligned} \tag{10}$$

where $\alpha \in [0, 1]$. Obviously, the firm will only remove a link if this action increases its profits.¹¹ Using the above notation we can give the following definition of a pairwise stable network.¹²

Definition 1. The graph G is pairwise stable if

- (i) $\forall ij \in E(G), \pi_i(G) \geq \pi_i(G-ij)$ and $\pi_j(G) \geq \pi_j(G-ij)$,
- (ii) $\forall ij \notin E(G)$, if $\pi_i(G+ij) > \pi_i(G)$ then $\pi_j(G+ij) < \pi_j(G)$, and, if $\pi_j(G+ij) > \pi_j(G)$ then $\pi_i(G+ij) < \pi_i(G)$.

From a straightforward application of the properties of the marginal revenues from collaboration (cf. item (iii) in [Lemma 1](#)) it follows that, on the one hand, when marginal costs are zero, $c=0$, links will always be created and no existing link will be deleted. The unique equilibrium is then the complete graph K_n , as stated in the next proposition.

Proposition 3. If costs are zero, $c=0$, then the complete graph K_n is the unique stable network.

On the other hand, when the difference between marginal costs c and severance costs $\nu(c)$ is larger than one, it is profitable to remove any link and the only equilibrium is the empty graph \bar{K}_n .

Proposition 4. For cost $c' = \alpha c > 1$ the empty graph \bar{K}_n is the unique stable network.

Besides the foregoing extreme situations, the determination of stable networks becomes quite involved. This is because, in general, the marginal revenue from a collaboration depends on the complete *topology* of the graph. In addition, for a given topology, it varies with the *position* of the firm which is chosen to create or delete a link (as discussed in [Section 2.2](#)). Starting from an initial graph $G(0)$, these properties imply that different network trajectories $G(1), G(2), \dots, G(t), \dots$ can be explored, according to the particular pair of firms that is allowed to revise its collaboration strategy at each time period $t \geq 1$. Thus, improving paths have a strong path dependent character in this model and multiple equilibrium networks might be possible for the same level of marginal costs. In what follows, we show that, multiple pairwise stable networks exist for the same value of marginal cost $c \in (0, 1)$ and severance costs $\nu(c)$.

We first show that a set of disconnected cliques of the same size can be a stable network, if their size falls within a certain interval that depends on the marginal cost of collaboration c and on the severance cost parameter α .¹³

¹¹ The level of severance costs affects the likelihood of link deletion in the model. In turn, the removal of a link in the network produces a negative externality on all firms in the same connected component as it reduces the size of the largest eigenvalue and thus of the benefit term (cf. [Lemma 1](#)). However, the “size” of this negative externality is the same for any level of severance costs, as it depends only on the size of the change in the largest eigenvalue following the link removal.

¹² Note that any network that is pairwise stable without severance cost is also pairwise stable with severance cost.

¹³ In the following, $\lceil x \rceil$, where x is a real valued number $x \in \mathbb{R}$, denotes the smallest integer larger or equal than x (the ceiling of x). Similarly, $\lfloor x \rfloor$ the largest integer smaller or equal than x (the floor of x).

Proposition 5. Consider costs c , $c' = \alpha c$ and $\alpha \in [0, 1]$. If the network G consists of a set of k equally sized, disconnected cliques $K_n^1, K_n^2, \dots, K_n^k$ (G having kn nodes in total) then G is stable if

$$\left\lceil \frac{1 + c(1 - c)}{c} \right\rceil \leq n \leq \left\lfloor \frac{2 - c'(1 - c')}{c'} \right\rfloor. \quad (11)$$

From Proposition 5 it immediately follows that for a given value of cost c there exist multiple integer values n (the size of the clique) that fit into the interval spanned by the upper and lower bounds in Eq. (5). This is discussed in more details in the proof of Proposition 5 (see Appendix A) and implies that multiple equilibrium networks exist for a given value of marginal cost c and severance cost $\nu(c)$.

Moreover, note that the homogeneous size of the cliques is a sufficient condition for stability and it allows for a simpler analytical treatment. However, it is not also a necessary condition. In the next section we show that, the equilibrium networks obtained with computer simulations show the existence of equilibria with disconnected cliques of different sizes (see e.g. Fig. 4, bottom-right). The requirement of having cliques of the same size appears in Proposition 5 only to simplify our analysis.

Equally sized disconnected cliques are not the only possible stable networks structures in the interval $c \in (0, 1)$ and $\alpha \in [0, 1]$. The next proposition shows that the spanning star, i.e. the star encompassing all nodes, can be pairwise stable as well, if the size of the star (and therewith the number of firms in the industry) falls within a certain region that depends on the cost c and on the severance cost parameter α .

Proposition 6. Consider costs c , $c' = \alpha c$, $\alpha \in [0, 1]$. The network G consisting of a spanning star $K_{1,n-1}$ with $\lceil 2/c \rceil \leq n \leq \lfloor (1 - c^2(6 + c^2))/4c^2 \rfloor$ is stable.

It should be noted at this point that these results characterize only some of the possible stable networks and that many other stable networks exist. As we show in Section 3.2 by means of computer simulations, there are various other possible equilibria with a rich, heterogeneous structure.

The foregoing results have two important implications in relation to the literature. First, stable graphs are not necessarily connected. Second, in general they are not minimally connected (see the definition in Section 2.1). Indeed, the multiple clique equilibrium is a disconnected graph in which each component is complete and thus not minimally connected. This is an important feature that for instance distinguishes our model from the “connections” model in Jackson and Wolinsky (1996) and from the linear “two-way flow” model in Bala and Goyal (2000). In both such models, the equilibrium networks are always connected, while in the latter they are also minimally connected. Furthermore, both models find that the spanning star is stable for intermediate values of the cost of collaboration. However, differently from both models, in our model the spanning star is *never* the unique stable network. Indeed, the next proposition combines the results of the previous two propositions, the conditions under which the link formation dynamics given in Definition 2 may lead to two different pairwise stable network topologies for the same level of marginal cost c and severance cost parameter α , namely (i) the set of disconnected equally sized cliques or (ii) the spanning star.

Proposition 7. Consider costs c , $c' = \alpha c$, $\alpha \in [0, 1]$ and the network G with n nodes such that $\lceil 2/c \rceil \leq n \leq \lfloor (1 + c^2(6 + c^2))/4c^2 \rfloor$. If there exists an integer $k \leq n$, $\text{mod}(n, k) = 0$ such that $\lceil (1 + c(1 - c))/c \rceil \leq k \leq \lfloor (2 + c'(1 - c'))/c' \rfloor$ then G can be stable for at least two cases.

- (i) G consists of disconnected cliques K_k^1, \dots, K_k^d , $n = kd$ or
- (ii) G consists of a spanning star $K_{1,n-1}$.

There are at least two stable networks for the same level of marginal cost c (degenerate cost region).

The multiplicity of equilibria stated in the above proposition is illustrated in Fig. 2. The plot shows the number of different values of size of cliques when the configuration of multiple cliques and the spanning star are both stable. One can see that for smaller values of α the number of stable networks increases. Furthermore, Fig. 3 shows two examples of possible equilibrium networks obtained with $n = 20$, $c = 0.3$ and $\alpha = 0.1$.

3.2. Topological properties of stable networks

The empirical research on R&D partnerships has investigated in depth the topological patterns of networks of knowledge exchange. From this literature (see e.g. Ahuja, 2000; Fleming et al., 2007; Hanaki et al., 2010; Powell et al., 2005), three features emerge as robust stylized facts: (i) R&D networks are sparse, that is the number of actual links is much less than the number of possible links. (ii) Networks are highly clustered, where clusters consist of highly interconnected firms, but different clusters are only sparsely connected. (iii) The distribution of links over the firms is characterized by high dispersion, with few firms being connected to many others.

The analytical study of equilibrium networks in Section 3 has pointed to the existence of equilibrium networks that match some of the stylized facts mentioned above. Indeed, equally sized cliques are characterized by high clustering, while the spanning star shows high degree heterogeneity. All these networks belong to the set of possible equilibrium structures in our model.

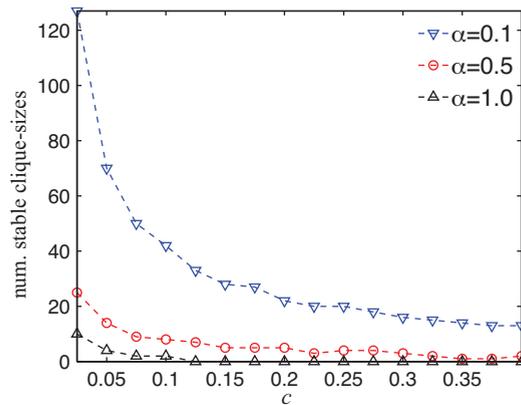


Fig. 2. Number of stable clique sizes when the spanning star $K_{1,n-1}$ is an equilibrium as well (for $\alpha = 0.1, 0.5$ and 1.0). If this number is positive then we have a spanning star $K_{1,n-1}$ and (at least one) set of disconnected cliques K_k^1, \dots, K_k^d as equilibrium networks for the same level of cost c .

In this section we define an explicit process of network evolution that is a particular case of an improving path. We analyze this process by means of computer simulations and identify the structural properties of stable networks in our model. In this way, we explore the existence of more complex stable network structures, beyond those described in the previous section. Furthermore, we investigate whether our model is also able to generate pairwise stable structures that feature, at the same time, all the stylized facts of R&D networks.

There are several possible processes which would be consistent with the definition of an improving path. In this work, we investigate a stochastic process in which all pairs of firms have the same probability to be selected to revise their R&D collaboration strategy (Jackson and Watts, 2002).

Definition 2 (Myopic pairwise dynamics). Let $G(t) = (N, E(t))$ denote the current graph at time $t \geq 0$. The initial graph $G(0)$ is the empty graph \bar{K}_n . We define the network formation process $(G(t))_{t \in \mathbb{N}}$ as follows. At the beginning of each period (at times $t = 0, 1, 2, \dots$) a single pair of firms, i and j , is uniformly selected at random from the set N of firms in $G(t)$.

- (i) If the link ij does not currently exist, $ij \notin E(t)$, then it is created whenever neither firm is harmed by the creation and at least one of them strictly gains, i.e.

$$\pi_i(G(t) + ij, c) \geq \pi_i(G(t), c) \wedge \pi_j(G(t) + ij, c) \geq \pi_j(G(t), c) \wedge \pi_i(G(t) + ij, c) > \pi_i(G(t), c) \vee \pi_j(G(t) + ij, c) > \pi_j(G(t), c). \tag{12}$$

- (ii) If the link ij is currently in place, $ij \in E(t)$, then it is removed whenever at least one of the firms strictly gains from the change, with link deletion involving the severance cost $\nu(c) = (1 - \alpha)c$, and $\alpha \in [0, 1]$. More formally

$$\pi_i(G(t) - ij, c, \nu) > \pi_i(G(t), c, \nu) \vee \pi_j(G(t) - ij, c, \nu) > \pi_j(G(t), c, \nu). \tag{13}$$

The above process of network evolution is a natural dynamic extension of the equilibrium notion of pairwise stability (cf. Definition 1). In addition, it has been already employed in dynamic analyses of strategic network formation (see Watts,

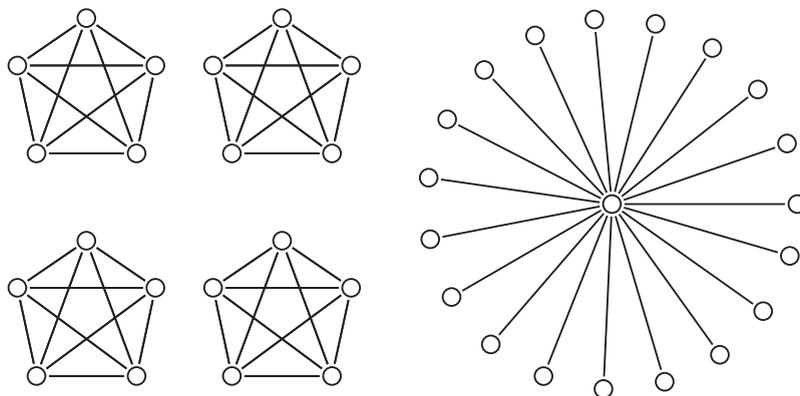


Fig. 3. An example of two possible equilibrium networks for cost $c = 0.3$ and $\alpha = 0.1$ with $n = 20$ firms. A set of disconnected cliques of the same size (left) and a spanning star (right).

2001; Vega-Redondo, 2007, p. 212). Finally, note that in the above process the only element of stochasticity is the sequence of the pairs of firms chosen to create or delete links.

We now study the stable network structures arising from this process in computational experiments conducted in a large region of the model's parameter space.¹⁴ Starting from an empty network $G(0) = \bar{K}_n$, each simulation is carried out according to the dynamics given in Definition 2, with a fixed industry size n , until the network is pairwise stable.¹⁵ The exact number of iterations needed to reach stability varies not only with the parameter set but also in each simulation, because of the path-dependent nature of the process. Typically, this is in the range of several thousands. For each given tuple of parameter values we carried out several runs in order to perform some statistics on the results. Both the marginal cost c and the severance cost parameter α vary in the interval $[0, 1]$. Various sets of simulations were performed. Those with largest network size had $n = 50$ and 50 runs per parameter set, and $n = 100$ and 10 runs. The figures in the following refer to the latter set.¹⁶ We did not observe any dependence of the stability results on the system size. Indeed, for the stability of the cliques (see Proposition 5) only the size of each connected component matter and not the total number of nodes. The results of the aforementioned Monte-Carlo experiments are shown in Figs. 4–6.

The plots in Fig. 4 show typical equilibrium networks obtained in simulations for marginal cost of link formation equal to 0.15 and different values of the severance cost parameter α . Recall that severance costs are equal to $v = (1 - \alpha)c$ and thus are inversely related to the parameter α . As the plots reveal, in this region of the parameter space the dynamics in our model is able to generate equilibrium structures displaying the complex features that characterize empirically observed R&D networks (see e.g. Fleming et al., 2007; Hanaki et al., 2010). In particular, for very high severance costs the equilibrium network contains a giant component with a high degree heterogeneity. On the other hand, as severance costs associated with link deletion fall (with increasing values of α), we observe a significant increase in the cliquishness of the network, and a reduction of degree heterogeneity.

The insights coming from the foregoing qualitative study are confirmed by a more quantitative analysis of the topological properties of equilibrium graphs. The plots in Fig. 5 display, respectively, the mean and the variance of the network degree distribution as functions of the marginal cost c and severance cost parameter α . The mean degree is inversely related to the sparseness of the graph, while degree variance captures degree heterogeneity (cf. stylized facts (i) and (iii), on p. 17). As the plots in the figure make clear, higher cost of R&D collaboration lead to graphs that are more sparse. On the other hand, degree heterogeneity reaches a peak for values of marginal cost close to 0.1, and then falls as collaboration costs increase. In addition, degree heterogeneity increases with severance costs (decreasing α). Notice that in our link formation mechanism we have not assumed any type of preferential attachment. Links are formed or deleted based solely on the marginal profit. Nonetheless, in many cases, nodes gain a larger increase in profit by forming a link to a hub rather than to a peripheral node.

The presence of clusters of highly interconnected firms is a key feature of empirically observed R&D networks (cf. stylized fact number (ii), on p. 17). As the plots in Fig. 6 show, this feature is also a characteristic of the equilibrium networks generated by the model. In particular, the average clustering coefficient (Fig. 6, top-left) is close to one in a wide region of the explored parameter space ($c \in (0, 0.5)$, $\alpha > 0$). This is due to the fact that many of the equilibrium networks consist of multiple cliques where all neighbors of any node are connected. As the cost increases, the cliques become smaller but their clustering remains high. Only when the size of the cliques drops from 3 to 2 nodes, then the clustering naturally drops to zero. Moreover, for a given cost, higher severance cost $v = (1 - \alpha)c$, i.e. lower α , leads to equilibrium networks which tend to depart more strongly from the multiple clique architecture and thus have smaller clustering. Finally, note that clustering becomes zero for values of costs greater or equal to 0.7.

Further insights into the topological features of R&D clusters can be obtained by looking at the average number of connected components, their average size and size concentration (Fig. 6, top-right, bottom-left and bottom-right respectively). As the plots in the figure indicate, the number of connected components is an increasing function of collaboration costs while the average size and its concentration are negatively related to collaboration costs. In addition, as the cost of link severance increases, the number of components also increases, while component size and size concentration decrease. For small values of c , $\langle N_H \rangle$ is close to one which implies that there is one dominant component. For increasing c , the size of the largest component sharply decreases, since the average number of the connected components $\langle N_H \rangle$ increases and the average Herfindal index of components size concentration (h_H) falls. The effect of the severance cost is that, as α increases, the size of the largest component decreases faster with c .

Joining together the foregoing results, we can conclude that sparse equilibrium networks organized in clusters of highly interconnected firms are a distinctive feature of the network dynamics in our model. Moreover, low values of R&D collaboration costs and high values of the costs of link severance lead to equilibrium structures characterized by a small number of large components, with a highly dispersed degree distribution. As collaboration costs increase and as link severance costs

¹⁴ When simulating the network evolution discussed in Section 3, the largest real eigenvalue of the network has to be computed. Since the largest real eigenvalue of a graph can be computed in polynomial time (Hong, 1993), our model is well suited for numerical investigations.

¹⁵ The criterion for convergence is exactly the definition of pairwise stable networks. This means that, repeatedly, the marginal payoff all pairs of nodes in the network has been computed until no pair of agents has an incentive to create or remove a link.

¹⁶ Because the computation of the marginal revenue from a link, is performed at every iteration for all the pairs (to check for stability), these simulations are computationally demanding and it was not possible to investigate systematically networks of larger size.

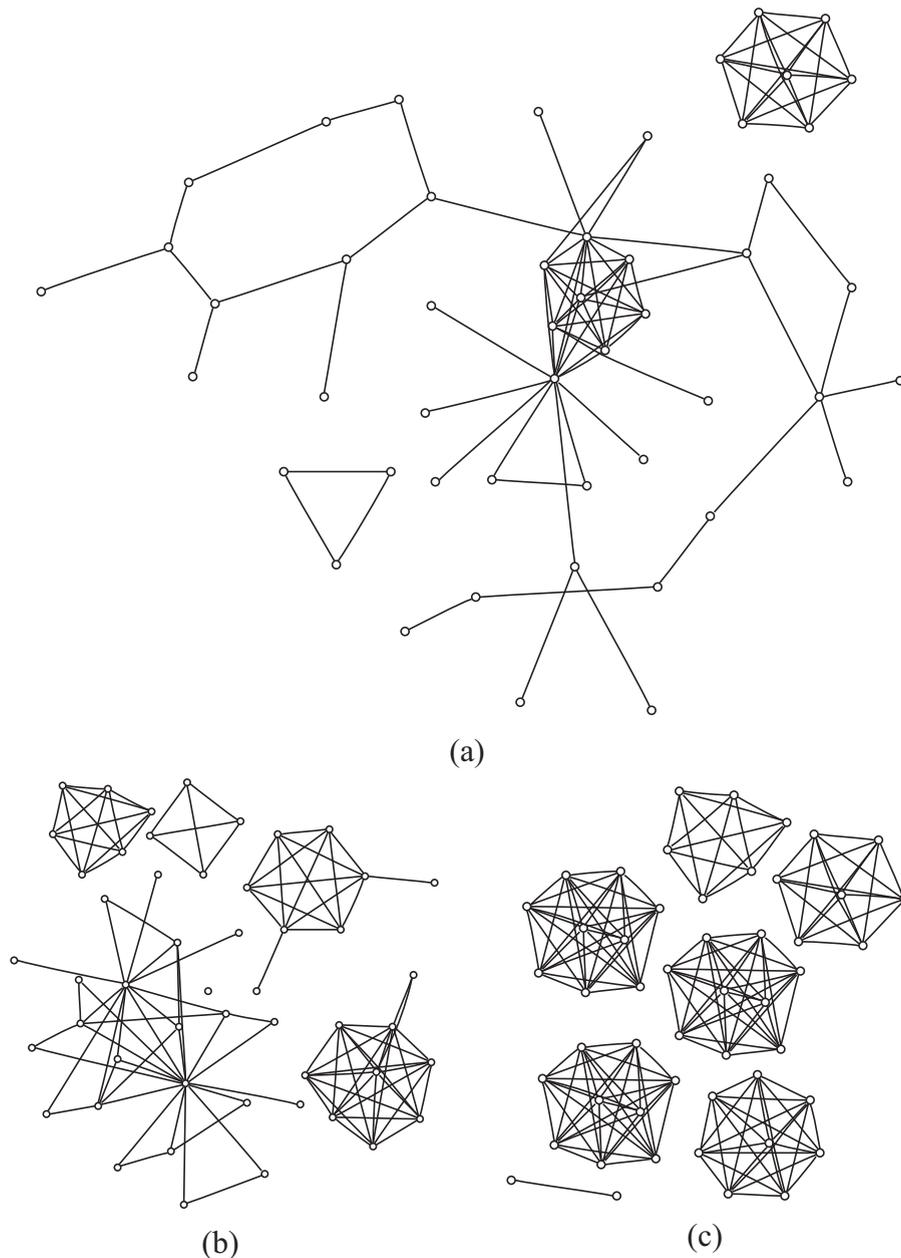


Fig. 4. Equilibrium networks for $n = 50$, $c = 0.15$, (a) $\alpha = 0.0$, (b) $\alpha = 0.1$, and (c) $\alpha = 1.0$ starting from an empty network.

decrease, we observe that equilibrium networks tend to be more and more organized in size homogeneous cliques having only few connections between them.

Finally, it is interesting to look at the performance (in terms of innovative capacity and aggregate profits) of the network structures discussed above. The plots in Fig. 7 show respectively the eigenvalue associated with the largest component in the network (on the left), and average profits (on the right), both taken as a function of c and α . The former is a measure of innovative capacity as it maps the highest possible knowledge (and innovation) growth rate in the industry, while the latter measures the overall performance of the industry. The plots show quite starkly that both variables decrease with the level of collaboration costs, whereas severance costs have a negligible effect on their levels.¹⁷ It turns out that both

¹⁷ The small effect of α on performance indicators can be explained by the fact that severance costs have a strong influence only on degree variance and on component size concentration. On the other hand, the level of the largest eigenvalue (and as a consequence average profits) depends on the density of links in the network (see Lemma 1), which is much more related to collaboration costs than to severance costs (cf. Fig. 5, left, and discussion above).

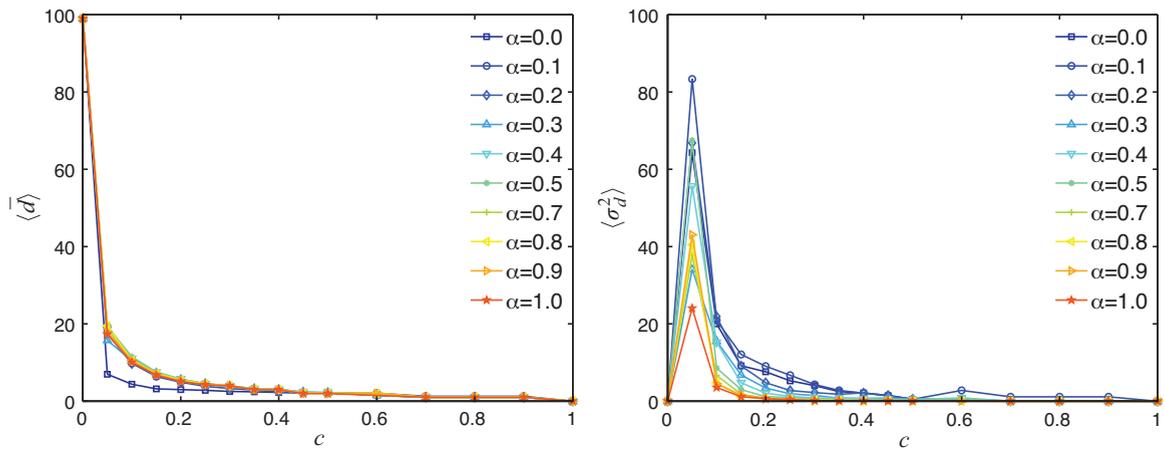


Fig. 5. Average degree (\bar{d}) and degree variance (σ_d^2) in the equilibrium network for $n=100$, $c \in [0, 1]$, starting from an empty network (averaged over 10 simulations).

innovative and overall performance of the industry are reached for low levels of the collaboration costs. In that region of the parameter space, networks are characterized by a strong level of connectedness (see above). These results have also implications for policies aimed at fostering research collaboration, such as for example the EU Framework Programs (EU FP). In particular, our results go in favor of policies aimed at increasing the level the cohesion across research units (e.g. the EU FP6 goal of creating a “European Research Area”). European policies conducted so far have indeed resulted in an increase of the connectedness of the network of research collaboration in Europe (see Breschi et al., 2009; Breschi and Cusmano, 2004, for an evaluation of EU FP programs from a network perspective). At the same time, our results indicate that further efforts should go towards the removal of the existing barriers to collaboration, like for example those associated with late entry in

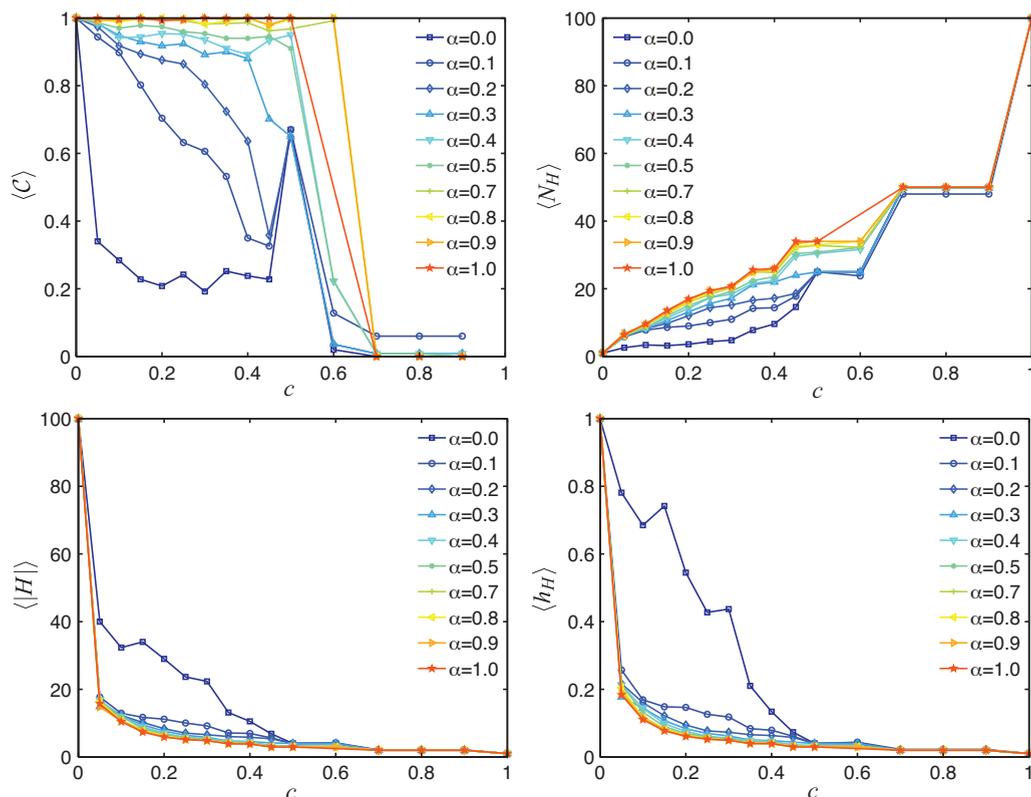


Fig. 6. Average clustering coefficient (C) (top, left), average number of components (N_H) (top, right), average size of components ($|H|$) (bottom, right) and average Herfindal index of components size concentration (h_H) (bottom, right) in the equilibrium network for $n=100$, $c \in [0, 1]$, starting from an empty network (averaged over 10 simulations).

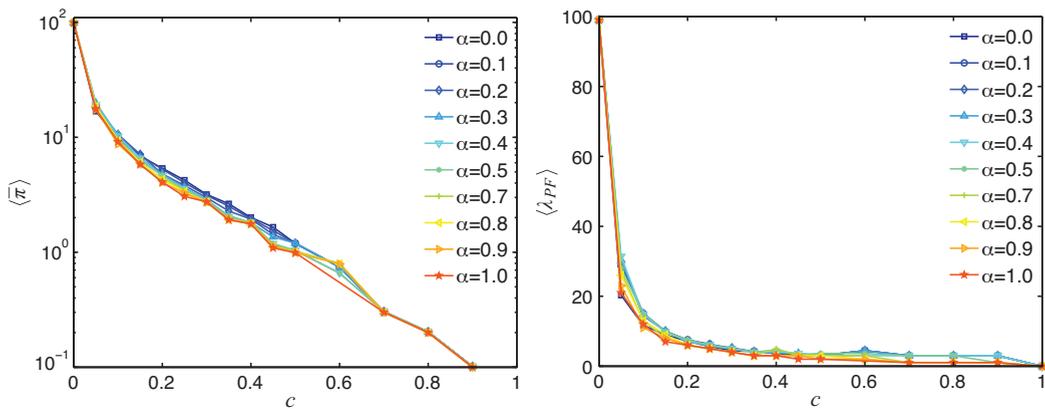


Fig. 7. Average profits $\bar{\pi}$ and the largest real eigenvalue λ_{PF} of equilibrium networks for $n = 100$, $c \in [0, 1]$ and $\alpha \in [0, 1]$, starting from an empty network (averaged over 10 simulations).

the project network (see [Breschi and Cusmano, 2004](#)), or related to spatial distance across units (e.g. [Hoekman et al., 2010](#)). The high collaboration costs implied by these barriers can indeed lead to a lock-in of the existing clusters of collaborations with a consequent breakdown of network connectedness, ultimately resulting in lower rates of knowledge recombination and innovative activity.

4. Conclusions

In this paper we investigated the evolution of networks of knowledge exchange across firms. We developed a model in which firms sharing common technological competencies recombine their knowledge stocks through R&D collaborations, in order to introduce innovations in the market. Since each collaboration is costly for firms they face a trade-off between the benefits of new collaborations (in terms of an increase in the number of innovations per period) and the costs associated with them. Furthermore, we showed that under mild conditions on the horizon over which the performance of R&D collaborations is evaluated, the benefit the firm receives from the network depends on all walks existing across firms in their connected component.

We studied the existence of pairwise stable networks (cf. [Jackson and Watts, 2002](#)), where we assumed that the deletion of an existing collaboration involves a severance cost. Severance costs capture the idea that collaborations embedded in a network are characterized by strong levels of reciprocity, and by mechanisms sanctioning opportunism by a partner ([Gulati, 1998](#)).

Following the notion of improving paths, we identified regions of collaboration and severance costs in which pairwise stable networks exist. We also showed that different network structures are stable for the same level of marginal costs. In particular, we identified regions of the collaboration and severance costs in which (i) the spanning star (i.e. the star encompassing all firms in the network), and (ii) disconnected cliques of equal size are stable.

The source of this multiplicity of equilibria lies in (i) the strong path dependency involved in partner selection decisions, (ii) in the presence of external effects affecting marginal revenue of collaborations for firms belonging to the same connected component, and (iii) in the inertia arising from the presence of a severance cost associated with the termination of a collaboration.

Finally, we investigated the topological characteristics of pair wise stable graphs in our model, to see whether they are able to replicate the stylized facts of empirically observed R&D networks. To this end, we studied via computer simulations the properties of equilibria generated under a two-sided myopic pairwise dynamics ([Jackson and Watts, 2002](#)). The results of our simulations show the existence of a region of low marginal costs of collaboration and high costs of link deletion in which the aforementioned dynamics are able to select pairwise stable structures matching the stylized facts of R&D networks. Our results have also implications for technology policy (e.g. European Research Programs). In particular, they bring support to policies enhancing network connectedness, as these networks in our model are characterized by higher aggregate performance than disconnected structures (the cliques).

The present work could be extended in several ways. First, the model could be extended to account for industry demand, for example like in [Goyal and Moraga-Gonzalez \(2001\)](#). In this way, one could then study how the network structure may change when firms operate in markets that are interdependent. Second, one could investigate whether the foregoing results about the properties of stable networks are robust to different link updating algorithms. For example, one could study the effect on the network dynamics of introducing firms pursuing heterogeneous strategies, for instance of the kind explored in [Bala and Goyal \(2000\)](#). Similarly, one could depart from the strong assumptions we made on the knowledge firms have about the network. In this respect, one could study instead the emergence of network structures when firms follow more

simple rules of behavior, for example of the kind suggested in the empirical work by Powell et al. (2005). Third, although our model generates a significant degree variance, we have not studied the higher moments of the degree distribution. One could therefore further develop the model in order to understand the conditions for the emergence of fat tails in the degree distribution (see e.g. Powell et al., 2005).

Finally, motivated by the evidence of technological convergence in R&D networks industries, we did not consider “local” barriers to technology exchange (e.g. due to technological distance across two partners) and we rather focused on “global” barriers, indirectly captured by the unitary collaboration costs. However, technological convergence and the increasing abstract and general character of the technology base in these industries, does not mean that tacitness and local technological complementarities (e.g. Arora and Gambardella, 1994a, 1994b; Foray, 2004) are not important in the process of knowledge exchange across firms. These two elements are likely to shape the ability of firms to evaluate and utilize the knowledge received from other firms, ultimately influencing the benefit reaped in a collaboration (see e.g. Arora and Gambardella, 1994b). A further analysis of R&D network dynamics should therefore embed all the foregoing ingredients related to industry technology, and try to investigate how they may affect the revenues and costs of the process of knowledge recombination.

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Appendix A.

In this appendix we give the proofs of the propositions and lemmas stated in the preceding sections.

Proof of Proposition 1.

(i) Since the matrix $\mathbf{A} + \gamma\mathbf{I}$ is diagonalizable, the general solution of Eq. (2) can be written as (Zwillinger, 1998):

$$\mathbf{X}(T) = \sum_{j=1}^n c_j \mathbf{v}_j e^{(\lambda_j + \gamma)T}, \quad (\text{A.1})$$

where c_i are unknown constants that are determined by the initial values $\mathbf{x}(0) = \sum_{j=1}^n c_j \mathbf{v}_j$, $\lambda_{\text{PF}} = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are the real eigenvalues of \mathbf{A} and $\mathbf{v}_1, \dots, \mathbf{v}_n$ the corresponding eigenvectors. Note that the addition of γ on the diagonal of \mathbf{A} simply shifts the spectrum of \mathbf{A} by γ . In Eq. (A.1) only those eigenvalues and corresponding eigenvectors of the adjacency matrix of the connected component G_i of firm i appear. All other eigenvalues have vanishing eigenvector components and do not contribute to the trajectory. This is intuitively clear since firms in disconnected components have decoupled equations of the form (2) and their trajectories can be computed independently.

Denoting by v_{ij} the i th component of the j th eigenvector we can write

$$\begin{aligned} \lambda_{\text{PF}} + \gamma - \frac{\dot{x}_i(\tau)}{x_i(\tau)} &= \frac{(\lambda_{\text{PF}} + \gamma)x_i(\tau) - \dot{x}_i(\tau)}{x_i(\tau)}, \\ &= \frac{\sum_{j=1}^n c_j v_{ij} e^{(\lambda_j + \gamma)\tau} (\lambda_{\text{PF}} - \lambda_j)}{x_i(\tau)}, \\ &= \frac{\sum_{j=2}^n c_j v_{ij} e^{(\lambda_j + \gamma)\tau} (\lambda_{\text{PF}} - \lambda_j)}{\sum_{j=1}^n c_j v_{ij} e^{(\lambda_j + \gamma)\tau}}. \end{aligned} \quad (\text{A.2})$$

In the numerator of Eq. (A.2) we obtain a sum of exponentials with one exponential term less than in the denominator, namely the one with the largest real eigenvalue in the exponent. We have that the sum of exponentials converges to the exponential with the largest real eigenvalue. For instance let $a \neq 0$, $\lambda_1 > \lambda_2$ then $a e^{\lambda_1 \tau} + b e^{\lambda_2 \tau} \approx a e^{\lambda_1 \tau}$ for large τ . Thus we get

$$\lambda_{\text{PE}} + \gamma \lim_{\tau \rightarrow \infty} \frac{\dot{x}_i(\tau)}{x_i(\tau)} = \lim_{\tau \rightarrow \infty} \frac{c_2 v_{i2} e^{(\lambda_2 + \gamma)\tau} (\lambda_{\text{PF}} - \lambda_2)}{c_1 v_{i1} e^{(\lambda_{\text{PF}} + \gamma)\tau}} \lim_{\tau \rightarrow \infty} e^{-(\lambda_{\text{PF}} - \lambda_2)\tau} = 0. \quad (\text{A.3})$$

Next, we compute a lower bound for the difference $\lambda_{PF} - \lambda_2$ and, therewith, the order of convergence. Consider the real eigenvalues $\lambda_{PF} = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ of the adjacency matrix \mathbf{A} . We have that $\sum_{j=1}^n \lambda_j^2 = \text{tr}(\mathbf{A}^2) = 2m$ (Bollobas, 1998). Thus, we get

$$\lambda_2^2 = 2m - \lambda_{PF}^2 - \sum_{j=3}^n \lambda_j^2 \leq 2m - \lambda_{PF}^2 \leq 2m - \left(\frac{2m}{n}\right)^2 = \frac{2m(n^2 - 2m)}{n^2}.$$

Here we have used the fact that $\lambda_{PF} \geq 2m/n$ (Bollobas, 1998). Therefore we get

$$\lambda_{PF} - \lambda_2 \geq \frac{2m - \sqrt{2m(n^2 - 2m)}}{n}, \tag{A.4}$$

which is positive and a monotonically increasing function for $n^2/4 < m \leq n(n-1)/2$. We thus have a fast convergence for dense networks. We note however, that in many cases the largest real eigenvalue is well separated from the second largest real eigenvalue. For example, in a random graph $G(n, p)$, the largest real eigenvalue grows with n (keeping p constant) while the upper bound on the second largest real eigenvalue grows with \sqrt{n} (Chung et al., 2003; Restrepo et al., 2007).

(ii) We show that the knowledge shares converge to the eigenvector \mathbf{v} associated with the largest real eigenvalue λ_{PF} of \mathbf{A} . The relative values of knowledge (shares) are given by $y_i = x_i / \sum_{j=1}^n x_j$ with $\sum_{j=1}^n y_j = 1$. Rewriting Eq. (2) by means of shares gives us the dynamics of the shares of knowledge

$$\dot{y}_i = \sum_{j=1}^n a_{ji} y_j - y_i \sum_{k,j=1}^n a_{jk} y_j. \tag{A.5}$$

Eq. (A.5) preserves the normalization of \mathbf{y} . If we consider a normalized eigenvector $\mathbf{y}^{(\lambda)}$, $\sum_{j=1}^n y_j^{(\lambda)} = 1$, associated with an eigenvalue λ of \mathbf{A} , we have

$$\sum_{j=1}^n a_{ij} y_j^{(\lambda)} = \sum_{j=1}^n a_{ji} y_j^{(\lambda)} = \lambda y_i^{(\lambda)}. \tag{A.6}$$

Inserting $\mathbf{y}^{(\lambda)}$ into Eq. (A.5) yields

$$\dot{y}_i = \sum_{j=1}^n a_{ji} y_j^{(\lambda)} - y_i^{(\lambda)} \sum_{k,j=1}^n a_{jk} y_j^{(\lambda)} = \lambda y_i^{(\lambda)} - y_i^{(\lambda)} \underbrace{\sum_{k,j=1}^n a_{jk} y_j^{(\lambda)}}_{\lambda \sum_{k=1}^n y_k^{(\lambda)} = \lambda} = 0.$$

Thus, $\mathbf{y}^{(\lambda)}$ is a stationary solution of Eq. (A.5). It can be shown that that an eigenvector \mathbf{v} associated with the largest real eigenvalue λ_{PF} of \mathbf{A} is the stable fixed point of Eq. (A.5)¹⁸ (see also Jain and Krishna, 1998; Zwillinger, 1998).

This concludes the proof. □

Proof of Proposition 2. We assume that $G(t + \tau) = G(t)$, $\tau \geq 0$, and that $G(t)$ is connected.¹⁹ Further, we make the substitution $\hat{x}_i(\tau) = x_i(t + \tau) / \sum_{j=1}^n x_j(t)$. The evolution of $\hat{x}_i(\tau)$ follows from Eq. (2) as²⁰

$$\frac{d\hat{x}_i(\tau)}{d\tau} = \gamma \hat{x}_i(\tau) + \sum_{j=1}^n a_{ij}(t) \hat{x}_j(\tau), \quad \hat{x}_i(0) = \frac{x_i(t)}{\sum_{j=1}^n x_j(t)} > 0, \tag{A.7}$$

for $i = 1, \dots, n$. Moreover, discounted profits of firm i from Eq. (6) become

$$\tilde{\pi}_i(G_i(t), \tilde{c}, t) = V \int_0^\infty \ln \hat{x}_i(\tau) e^{-\rho\tau} d\tau + \frac{V \sum_{j=1}^n x_j(t)}{\rho} - \left(\frac{\tilde{c}}{\rho^2} + \frac{\tilde{c}t}{\rho} \right) d_i(t). \tag{A.8}$$

¹⁸ If the largest real eigenvalue has multiplicity more than one then the stable fixed point can be written as a linear combination of the associated eigenvector and generalized eigenvectors (Braun, 1993).

¹⁹ Otherwise, we consider only the connected component $G_i(t)$ of firm i .

²⁰ We have that $x_i(0) > 0$ and $\dot{x}_i > 0$ so that $x_i(t) > 0$ for all $i = 1, \dots, n$ and $0 \leq t < \infty$. Further, $x_i(t)$ is the solution of a system of linear ordinary differential equations which yields a bounded solution for finite values of t . It follows that $\hat{x}_i(0) > 0$ for all $i = 1, \dots, n$.

Note that the term $(1/\rho)V\sum_{j=1}^n x_j(t)$ in Eq. (A.8) is not affected by the linking strategy of the firm (instead, it depends only on the stocks of knowledge of the firms at the time when a new link is created or an old one removed and not on the knowledge generated thereafter) and thus can be omitted from the profit function $\tilde{\pi}_i(G_i(t), \tilde{c}, t)$ when assessing the marginal profit of a link. We can write the solution of Eq. (A.7) as $\hat{\mathbf{x}}(\tau) = \sum_{j=1}^n c_j \mathbf{v}_j e^{(\lambda_j + \gamma)\tau}$, where c_i are unknown constants that are uniquely determined by the initial condition $\hat{\mathbf{x}}(0) = x_i(t)/\sum_{j=1}^n x_j(t) = \sum_{j=1}^n c_j \mathbf{v}_j$ (Zwillinger, 1998, p. 383). By using this expression for $\hat{x}_i(\tau)$, we can write

$$\int_0^\infty \ln \hat{x}_i(\tau) e^{-\rho\tau} d\tau = \frac{\lambda_1 + \gamma}{\rho^2} + \int_0^\infty \ln \left[c_1(\mathbf{v}_1)_i + \sum_{j=2}^n c_j(\mathbf{v}_j)_i e^{-(\lambda_1 - \lambda_j)\tau} \right] e^{-\rho\tau} d\tau, \tag{A.9}$$

where $(\mathbf{v}_j)_i$ is the i th component of the j th eigenvector. Let

$$C = \max_{\tau \in \mathbb{R}_{\geq 0}} \left| \ln \left[c_1(\mathbf{v}_1)_i + \sum_{j=2}^n c_j(\mathbf{v}_j)_i e^{-(\lambda_1 - \lambda_j)\tau} \right] \right|. \tag{A.10}$$

In the following, we show that $C < \infty$ for all $\tau \in \mathbb{R}_{\geq 0}$, we have that

$$\ln \left[c_1(\mathbf{v}_1)_i + \sum_{j=2}^n c_j(\mathbf{v}_j)_i \right] = \ln[\hat{x}_i(0)] = \ln \left[\frac{x_i(t)}{\sum_{j=1}^n x_j(t)} \right].$$

Since $x_i(t) > 0$ and $x_j(t) < \infty$ for $t < \infty$ and any $i, j = 1, \dots, n$, we must have that $0 < x_i(t)/\sum_{j=1}^n x_j(t) < t$, and the above expression is finite. Next, consider $0 < \tau < \infty$. We have that

$$\ln \left[c_1(\mathbf{v}_1)_i + \sum_{j=1}^n c_j(\mathbf{v}_j)_i e^{-(\lambda_1 - \lambda_j)\tau} \right] = \ln \left[e^{-(\lambda_1 + \gamma)\tau} \left(\sum_{j=1}^n c_j(\mathbf{v}_j)_i e^{(\lambda_j + \gamma)\tau} \right) \right] = [\ln e^{-(\lambda_1 + \gamma)\tau} \hat{x}_i(\tau)]. \tag{A.11}$$

First, notice that $\sum_{j=1}^n c_j(\mathbf{v}_j)_i e^{-(\lambda_1 - \lambda_j)\tau} \leq \sum_{j=1}^n |c_j| |(\mathbf{v}_j)_i| \leq n$. The last inequality follows from the initial condition $\mathbf{x}(0) = \sum_{j=1}^n c_j \mathbf{v}_j$, and multiplying with \mathbf{v}_j yields $c_j = \mathbf{v}_j \hat{\mathbf{x}}(0)$, for $j = 1, \dots, n$, and $\sum_{i=1}^n \hat{x}_i(0) = 1$. Thus, c_j is a convex combination of the normalized eigenvector \mathbf{v}_j and therefore $|c_j| \leq 1$.²¹ Second, with for any $0 < \tau < \infty$ the argument of the logarithm is positive. Therefore, the above expression is finite for $0 < \tau < \infty$. Finally, consider the limit $\tau \rightarrow \infty$. Then

$$\lim_{\tau \rightarrow \infty} \left[c_1(\mathbf{v}_1)_i + \sum_{j=2}^n c_j(\mathbf{v}_j)_i e^{-(\lambda_1 - \lambda_j)\tau} \right] = \ln[c_1(\mathbf{v}_1)_i].$$

Note that \mathbf{v}_1 is a strictly positive eigenvector. Further, c_1 must be positive. For contradiction, assume that $c_1 = 0$. Then $\hat{\mathbf{x}}(0)$ would be orthogonal to \mathbf{v}_1 and $\mathbf{v}_1 \hat{\mathbf{x}}(0) = 0$. However, this is not possible for both $\hat{\mathbf{x}}(0) > 0$ and $\mathbf{v}_1 > 0$. Moreover, $c_1 < 0$ would imply that $\lim_{\tau \rightarrow \infty} \hat{x}_i(\tau) < 0$. Also, this is a contradiction. Therefore, we must have that $c_1 > 0$. This implies that $|\ln[c_1(\mathbf{v}_1)_i]| < \infty$.

It follows that

$$\left| \int_0^\infty \ln \left[c_1(\mathbf{v}_1)_i + \sum_{j=2}^n c_j(\mathbf{v}_j)_i e^{-(\lambda_1 - \lambda_j)\tau} \right] e^{-\rho\tau} d\tau \right| \leq \frac{C}{\rho}, \tag{A.12}$$

and therefore

$$\int_0^\infty \ln \hat{x}_i(\tau) e^{-\rho\tau} d\tau = \frac{\lambda_1 + \gamma}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho}\right).$$

Inserting this into Eq. (A.8), the desired claim follows. \square

Proof of Proposition 3. If costs are zero, $c=0$, then the change in eigenvalue equals the change in profits. Since (in a connected graph G) each link created strictly increases λ_{pf} (Horn and Johnson, 1990) and accordingly profits, the complete graph K_n is reached eventually. \square

²¹ Since \mathbf{v}_j is normalized we have that $(\mathbf{v}_j)_i \leq 1$ and it follows that $c_i = \sum_{j=1}^n (\mathbf{v}_i)_j \hat{x}_j(0) \leq \sum_{j=1}^n \hat{x}_j(0) = 1$ for both $\mathbf{v}_i, \hat{\mathbf{x}}(0)$ positive and all i, j, \dots, n .

Proof of Proposition 4. There exists the following bound on the change in eigenvalue by the removal and creation of a link (Cvetkovic et al., 1995): if the graphs G, G' differ in one edge only then $|\lambda_{\text{PF}}(G') - \lambda_{\text{PF}}(G)| \leq 1$. A link is created if $\Delta\lambda_{\text{PF}} > c$. Thus, no link is created if $c = 1$. On the other hand, a link is removed, if $\Delta\lambda_{\text{PF}} > c'$. And thus, all links are removed if $c' > 1$ and we obtain an empty graph K_n . \square

Proof of Proposition 5. We want to show that the graph G consisting of k cliques of equal size is stable, that is, no link is removed or created. For the removal, we can focus on links between nodes in the same clique, since these are the only links in G . Thus, in Proposition 8 we show that, for any pair of nodes in the same clique, a link is not removed as long as the size n of the clique is smaller than a given bound b_r . In particular, we show that, $b_r = [(2 - c'(1 - c'))/c']$.

For the creation of links, we can focus on links between nodes in different cliques, since these are the only new links that can be added to the graph. Thus, in Proposition 9 we show that for any pair of nodes belonging to different cliques, a link between them is not created as long as the size n of the clique is larger than another bound b_c , and we show that $b_c = [(1 + c(1 - c))/c]$.

It turns out that the bound for the removal, b_r , is larger than the bound for the creation, b_c , for any value of $c \in [0, 1]$. However, since the size n of the clique has to be an integer, the interval $[b_c, b_r]$ needs to contain at least one integer. This can be done constructively. We explored the interval $c \in [0, 1]$ with cost increments of 10^{-3} and we counted the number of integer values that fall within $[b_c, b_r]$. One can show that for $c < 0.35$, there is always at least one integer in between the two bounds, while for $c < 0.2$, there are always several integers falling in between the two bounds.

This is a remarkable finding as it implies that for the values of cost given above, there exists a multiplicity of equilibria. Indeed, for a given value of cost, the stable graphs are all the configurations with cliques of the same size n , where n varies among the integers included in the interval $[b_c, b_r]$.

Propositions 8 and 9, used for this proof, are given below. \square

Proposition 8. Consider a clique K_n and denote by $K_n - ij$ the graph obtained from K_n by removing an edge ij . Then $\lambda_{\text{PF}}(K_n) - \lambda_{\text{PF}}(K_n - ij) > c'$ if $n \leq \lfloor (2 - c'(1 - c'))/c' \rfloor$.

Proof of Proposition 8. Denote the matrix obtained from the adjacency matrix \mathbf{A} of $K_n - ij$, and subtracting the variable λ on the diagonal of \mathbf{A} by $\mathbf{M} = \mathbf{A} - \lambda \mathbf{I}$. \mathbf{M} is a block matrix of the form

$$\mathbf{M} = \begin{pmatrix} \mathbf{K} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{D} \end{pmatrix}, \tag{A.13}$$

with submatrices²²

$$\mathbf{K} = \begin{pmatrix} -\lambda & 1 & \dots & 1 \\ 1 & -\lambda & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & -\lambda \end{pmatrix}_{(n-2) \times (n-2)} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \end{pmatrix}_{2 \times n}^T \quad \mathbf{D} = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}_{2 \times 2}.$$

Since \mathbf{M} is a block-matrix (Horn and Johnson, 1990) we can write

$$\det(\mathbf{M}) = \det(\mathbf{K}) \det(\mathbf{P}). \tag{A.14}$$

For the determinant of \mathbf{K} we obtain

$$\det \mathbf{K} = -((n - 1) - \lambda)(1 + \lambda)^{n-1}. \tag{A.15}$$

The Schur complement is $\mathbf{P} = \mathbf{D} - \mathbf{B}\mathbf{K}^{-1}\mathbf{B}^T$. Multiplying the inverse of \mathbf{K} with \mathbf{B} from the left and \mathbf{B}^T from the right we obtain

$$\mathbf{B}\mathbf{K}^{-1}\mathbf{B}^T = \|\mathbf{K}^{-1}\|_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{A.16}$$

where $\|\mathbf{K}^{-1}\|_1$ is the sum of all elements in the matrix \mathbf{K}^{-1} (the l_1 -norm of the matrix \mathbf{K}^{-1} (Horn and Johnson, 1990)). By computing $\mathbf{K}^{-1}\mathbf{K} = \mathbf{I}$ one can verify that

$$\mathbf{K}^{-1} = \begin{pmatrix} \frac{n - 4 - \lambda}{(\lambda - (n - 3))(1 + \lambda)} & -\frac{1}{(\lambda - (n - 3))(1 + \lambda)} & \dots \\ -\frac{1}{(\lambda - (n - 3))(1 + \lambda)} & \ddots & \\ \vdots & & \end{pmatrix}. \tag{A.17}$$

²² The numbers at the bottom right of the matrix indicate the dimension of the matrix.

And, by summation over the elements in \mathbf{K}^{-1} , we obtain $\|\mathbf{K}^{-1}\|_1 = (n - 2/(n - 3) - \lambda)$. Consequently, the determinant of the Schur complement \mathbf{P} is given by

$$\det(\mathbf{P}) = (1 + \lambda)^{n-3} \lambda (\lambda^2 - (n - 3)\lambda - 2(n - 2)). \tag{A.18}$$

The largest real eigenvalue of $K_n - ij$

$$\lambda^2 - (n - 3)\lambda - 2(n - 2) = 0. \tag{A.19}$$

Thus we get

$$\lambda_{\text{PF}} = \frac{1}{2} \left(n - 3 + \sqrt{n^2 + 2n - 7} \right). \tag{A.20}$$

For the change in eigenvalue $\Delta\lambda_{\text{PF}} = \lambda_{\text{PF}}(K_n) - \lambda_{\text{PF}}(K_n - ij)$ we obtain

$$\Delta\lambda_{\text{PF}} = \frac{1}{2} \left((n + 1) - \sqrt{n^2 + 2n - 7} \right), \tag{A.21}$$

since $\lambda_{\text{PF}}(K_n) = n - 1$. This is a decreasing function in n . Then for $n \in \mathbb{N}$, $\Delta\lambda_{\text{PF}} > c'$ if

$$n \leq \left\lfloor \frac{2 - c'(1 - c')}{c'} \right\rfloor. \tag{A.22}$$

For $c' = 2 - \sqrt{2} = 0.586$ we have $n \leq 3$ and for $c = 1$ we obtain $n \leq 2$. \square

Proposition 9. Denote the graph consisting of two disconnected cliques by G and the graph obtained from G by connecting the two cliques in G via an edge by G' . Then for $n \geq \lceil (1 + c(1 - c))/c^2 \rceil$ we have $\lambda_{\text{PF}}(G') - \lambda_{\text{PF}}(G) < c$. \square

Proof of Proposition 9. Denote the adjacency matrix of the graph obtained by connecting two complete subgraphs K_n and K_n via an edge. And denote the matrix obtained by subtracting the variable λ on the diagonal of \mathbf{A} by $\mathbf{M} = \mathbf{A} - \lambda\mathbf{I}$. The eigenvalues of \mathbf{A} are given by the roots of the determinant of \mathbf{M} . \mathbf{M} has the form of a block matrix with the submatrices \mathbf{K} and \mathbf{B} . We have

$$\mathbf{M} = \begin{pmatrix} \mathbf{K} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{K} \end{pmatrix}_{2n \times 2n} \quad \mathbf{K} = \begin{pmatrix} -\lambda & 1 & \dots & 1 \\ 1 & -\lambda & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & -\lambda \end{pmatrix}_{n \times n} \quad \mathbf{B} = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & 0 & & \vdots \\ 0 & 1 & 0 & \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}_{n \times n},$$

w.l.o.g. we have put the one on the diagonal in \mathbf{B} indicating the link between the cliques. Then the Schur complement is given by

$$\mathbf{P} = \mathbf{K} - \mathbf{B}^T \mathbf{K}^{-1} \mathbf{B}. \tag{A.23}$$

For the determinant of \mathbf{M} we have $\det \mathbf{M} = \det(\mathbf{K}) \det(\mathbf{P})$. The determinant of \mathbf{K} is given by

$$\det(\mathbf{K}) = (1 + \lambda)^{n-3} (\lambda - n + 3). \tag{A.24}$$

The inverse of \mathbf{K} is already given in (A.17).²³ W.l.o.g. the Schur complement \mathbf{P} is given by

$$\mathbf{P} = \begin{pmatrix} -\lambda & 1 & \dots & \dots & 1 \\ 1 & -\lambda & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & -\lambda & 1 \\ 1 & \dots & 1 & -\lambda + \frac{\lambda - (n - 2)}{(\lambda - (n - 1)(1 + \lambda))} & \end{pmatrix}. \tag{A.25}$$

We can compute the determinant of the Schur complement as follows \mathbf{P}

$$\det \mathbf{P} = -(1 - n + (n - 2)q - \lambda q)(1 + \lambda)^{n-2},$$

$$q := \frac{-(n - 2) + \lambda}{(-(n - 1) + \lambda)(1 + \lambda)},$$

²³ Note that here the matrix \mathbf{K} has dimension $n \times n$.

λ_{PF} is given by the largest root of $\det \mathbf{P} = 0$. We obtain $\lambda_{PF} = (1/2) \left(n - 1 + \sqrt{n^2 - 2n + 5} \right)$. The change in the largest real eigenvalue is

$$\begin{aligned} \Delta \lambda_{PF} &= \frac{1}{2} \left(n - 1 + \sqrt{n^2 - 2n + 5} - (n - 1) \right), \\ &= \frac{1}{2} \left(1 - n + \sqrt{(n - 2)n + 5} \right), \end{aligned}$$

since $\lambda_{PF}(K_n) = n - 1$. Thus, $\Delta \lambda_{PF} < c$ if

$$n \geq \left\lceil \frac{1 + c(1 - c)}{c} \right\rceil. \quad (\text{A.26})$$

For costs $c = 0.5$ we get $n \geq 2$ and for $c = 1$ we get $n \geq 1$. \square

Proof of Proposition 6. In order to proof the stability of the spanning star $K_{1,n-1}$, that connects all nodes in the network, we have to consider two cases: (i) the creation of a link and (ii) the removal of a link.

(i) We consider the creation of a link between the nodes in the star. The normalized eigenvector \mathbf{v} associated with the largest real eigenvalue λ_{PF} is given by

$$\mathbf{v} = \frac{1}{\sqrt{2(n-1)}} (1, \dots, 1, \sqrt{n-1}, 1, \dots, 1)^T. \quad (\text{A.27})$$

Maas (1987) found an upper bound for the largest real eigenvalue λ_{PF} and the corresponding eigenvector \mathbf{v} of an undirected graph G if an edge ij is added

$$\lambda_{PF} = (G + ij) - \lambda_{PF}(G) < 1 + \delta - \frac{\delta(1 + \delta)(2 + \delta)}{(\vartheta_i + \vartheta_j)^2 \delta(2 + \delta + 2\vartheta_i\vartheta_j)}, \quad (\text{A.28})$$

where δ denotes the minimum degree in the graph G .²⁴ Applying Eq. (A.28) to the star $K_{1,n-1}$ gives $\Delta \lambda_{PF} = \lambda_{PF}(K_{1,n-1} + ij) - \lambda_{PF} < (K_{1,n-1}) < 2/n$. The link ij is not created if $\Delta \lambda_{PF} < c$ or equivalently $n > 2/c$. This is a decreasing function in c .

(ii) The change in eigenvalue by removing a link from $K_{1,n-1}$ is given by $\Delta \lambda_{PF} = \lambda_{PF}(K_{1,n-1}) - \lambda_{PF} < (K_{1,n-2}) = \sqrt{n-1} - \sqrt{n-2}$. A link is not removed from the star if $\Delta \lambda_{PF} > c'$ or equivalently

$$2 < n < \frac{1 + c'^2(6 + c'^2)}{4c'^2}. \quad (\text{A.29})$$

Putting the bounds obtained in (i) and (ii) together we get the desired proposition.

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²⁴ Eq. (A.28) is only an upper bound so that the number of stable stars may actually be higher.

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